Stochastic Analysis for Lagrangian Particles Simulation

Radu Maftei

École doctorale n° 364 : Sciences Fondamentales et Appliquées
Unité de Recherche : Inria Sophia Antipolis EPI TOSCA

Thèse de doctorat

Présentée en vue de l'obtention du grade de docteur en mathématiques de l'Université Côte d'Azur

par

Radu Maftei

Analyse stochastique pour la simulation de particules lagrangiennes

Application aux collisions de particules colloïdes

dirigée par Mme. Mireille Bossy et co-dirigé par M. Jean-Pierre Minier

Soutenue 14 décembre 2017, devant le jury composé de

M. Cédric Bernardin,	Professeur, Université de Nice Sophia-Antipolis	Examinateur
Mme. Mireille Bossy,	Directrice de Recherche, Inria	Directrice
M. Emmanuel Gobet,	Professeur, École Polytechnique	Examinateur
M. Benjamin Jourdain,	Professeur, École Nationale des Ponts et Chaussées	Rapporteur
M. Jean-Pierre Minier,	Chercheur R&D, EDF	Co-directeur
M. Fabien Panloup,	Professeur, Université d'Angers	Rapporteur
M. Christophe Profeta,	Maître de Conférences, Université d'Évry	Examinateur

Abstract

Cette thèse s'inscrit dans le cadre plus général de la simulation de particules colloïdales qui joue un rôle important dans la compréhension des écoulements diphasiques. Plus précisément, nous nous intéressons aux particules dans un écoulement turbulent et modélisons leur dynamique par un processus stochastique lagrangien, leurs interactions comme des collisions parfaitement élastiques où l'influence de l'écoulement est modélisée par un terme de force sur la composante vitesse du système. En couplant les particules deux par deux et considérant leurs position et vitesse relatives, la collision parfaitement élastique devient une condition de réflexion spéculaire. Nous proposons un schéma de discrétisation en temps pour le système Lagrangien résultant avec des conditions aux bords spéculaires et prouvons que l'erreur faible diminue au plus linéairement dans le pas de discrétisation temporelle. La démonstration s'appuie sur des résultats de régularité de l'EDP Feynman-Kac et requiert une certaine régularité sur le terme de force. Nous expérimentons numériquement certaines conjectures, dont l'erreur faible diminuant linéairement pour des termes de force qui ne respectent pas les conditions du théorème. Nous testons le taux de convergence de l'erreur faible pour l'extrapolation Richardson Romberg et le fait qu'un algorithme Multilevel Monte Carlo demeure efficace. Enfin, nous nous intéressons aux approximations Lagrangiennes/Browniennes en considérant un système Lagrangien où la composante vitesse se comporte comme un processus rapide. Nous contrôlons l'erreur faible entre la composante position du modèle Lagrangien et un processus de diffusion uniformément elliptique choisi de manière appropriée. Nous démontrons ensuite un contrôle similaire en introduisant une limite réfléchissante spéculaire sur le système Lagrangien et une réflexion appropriée sur la diffusion elliptique.

***Mots clés*—** Système Lagrangien, réflexion spéculaire, erreur faible, approximation Smoluchowski-Kramers

Abstract

This thesis broadly concerns colloidal particle simulation which plays an important role in understanding two-phase flows. More specifically, we track the particles inside a turbulent flow and model their dynamics as a stochastic process, their interactions as perfectly elastic collisions where the influence of the flow is modelled by a drift on the velocity term. By coupling each particle and considering their relative position and velocity, the perfectly elastic collision becomes a specular reflection condition. We put forward a time discretisation scheme for the resulting Lagrange system with specular boundary conditions and prove that the convergence rate of the weak error decreases at most linearly in the time discretisation step. The evidence is based on regularity results of the Feynman-Kac PDE and requires some regularity on the drift. We numerically experiment a series of conjectures, amongst which the weak error linearly decreasing for drifts that do not comply with the theorem conditions. We test the weak error convergence rate for a Richardson Romberg extrapolation and the effectiveness of the Multilevel Monte Carlo algorithm. We finally deal with Lagrangian/Brownian approximations by considering a Lagrangian system where the velocity component behaves as a fast process. We control the weak error between the position of the Lagrangian system and an appropriately chosen uniformly elliptic diffusion process and subsequently prove a similar control by introducing a specular reflecting boundary on the Lagrangian and an appropriate reflection on the elliptic diffusion.

***Keywords*—** Lagrangian system, specular reflection, weak error, Smoluchowski-Kramers approximation

Remerciements

La thèse de doctorat représente un travail qui s'inscrive dans la durée et marque la fin de la vie étudiante et l'entrée dans le monde scientifique. Ce travail ne pourrait être réalisé sans l'apport significatif sur beaucoup d'aspects - scientifique, professionnel, personnel, moral - et je tiens à remercier toutes les personnes qui m'ont aidé à accomplir ce travail.

Je souhaite tout d'abord exprimer mes plus vifs remerciements à ma directrice de thèse, Madame Mireille Bossy pour m'avoir accordé sa confiance et soutenu avec beaucoup de patience et et de persévérance au long de ces trois années afin de donner forme et mener à bien ce travail. Les différents commentaires, pistes de recherche à explorer, recommandations, ainsi que l'appui moral pour dépasser les difficultés spécifiques à une telle épreuve (surtout dans le domaine des mathématiques) ont représenté les repères sans lesquels cette thèse ne serait pas aboutie.

Mes plus sincères remerciements vont également à Monsieur Jean-Pierre Minier, chercheur R&D à l'EDF, et à Monsieur Christophe Profeta, maître de conférences à l'Université d'Évry, dont le soutien sans faille, la disponibilité et les conseils avisés m'ont permis de bien démarrer et finaliser la thèse.

Je suis très reconnaissant à Monsieur Jean-François Jabir, maître de conférences à l'Université de Valparaiso, pour son invitation de mener une partie de ma recherche au Chili et surtout pour les observations constructives et les réponses détaillées apportées avec gentillesse à tous mes questions. Son travail avec Mireille a eu une grande influence sur l'aboutissement du premier chapitre de cette thèse.

J'exprime ma profonde gratitude à Monsieur Benjamin Jourdain, professeur à l'ENPC et à Monsieur Fabien Panloup, professeur à l'Université d'Angers pour l'honneur qu'il m'ont fait en acceptant d'être rapporteurs de cette thèse, pour y avoir apporté une lecture attentive et pour les remarques précieuses qui ont permis d'améliorer sa qualité. Je tiens aussi à remercier Messieurs Cédric Bernardin, professeur à l'Université de Nice Sophia-Antipolis, Emmanuel Gobet, professeur à l'École Polytechnique et Christophe Profeta pour avoir accepté de faire partie de ce jury.

Je témoigne ma reconnaissance à Monsieur Denis Talay et aux autres membres permanents de l'équipe TOSCA, Monsieur Étienne Tanré et Madame Christine Faber pour m'avoir chaleureusement accueilli au sein de l'équipe et pour avoir su créer un environnent très convivial. Je remercie aussi Christine pour son aide, ses recommandations, sa grande disponibilité et sa patience concernant les tâches d'ordre administratif, la vie à l'INRIA, les interactions avec le centre ou les différents organismes.

J'aimerais aussi adresser une pensée sympathique aux doctorants, post-doctorants et stagiaires : Milica, Quentin, Pascal, Hector, Maxime, James et aux autres avec qui j'ai partagé énormément de moments d'amitié et qui ont su développer et entretenir une ambiance très agréable au sein de l'équipe, faisant que les situations difficiles qui accompagnent naturellement une thèse soient surmontées plus aisément.

Finalement, je remercie de tout cœur ma famille et mes amis d'avoir été là. Sans eux, je ne serais même pas arrivé à débuter ma thèse, encore moins la finir.

Contents

Introduction

Two-phase flows are flows of turbulent fluid which contain discrete elements: solid particles, droplets or bubbles. The properties of these flows can vary significantly as a function of the size or density of the elements and according to the different types of forces that influence their motion inside the flow. We shall consider the case where the random molecular fluctuations inside the fluid are more important than the gravitational and inertial effects. In order to achieve this condition the diameter of the discrete element d_p must be sufficiently small:

$$d_p \approx \left(\frac{\rho_f^2 \nu_f^2}{\rho_p^3 g^2} k_{\mathrm{B}} \Theta_f \right)^{\frac{1}{7}}$$

where ρ_f, ν_f, Θ_f are the density, fluid kinematic viscosity and respectively temperature of the fluid, ρ_p the density of the element, k_{B} Boltzmann's constant and g the gravitational acceleration. Under standard conditions for temperature and pressure (288.15 K and 1 atm), this implies that $d_p \leq 1\mu$m. The diameter also needs to be sufficiently large, of at least several nanometres in order to avoid quantum effects. Particles in this range, larger than a few hundred nanometres and smaller than a micrometre, are called colloidal particles. We shall also assume that they are solid, which allows tracking by only considering the center of gravity.

The modelling of two-phase flows with colloidal particles is important since it can describe the dispersion of aerosols or pollutants in the atmosphere, the agglomeration of radioactive particles in the steam turbines of nuclear reactors and many other situations. We shall specifically focus on colloidal particle collisions. Several approaches have been developed to model such collisions, of which we mention two complementary views:

- approaches based on the collision kernel modelling, defined formally as the collision rate divided by the concentration of particles.
- approaches based on direct particle tracking simulations where the particle dynamics and interactions are explicitly calculated. More details on these approaches can be found in [Henry *et al.*, 2014]

The first type of approach requires the knowledge of a collision kernel which is introduced in a population balance equation that can give the evolution of the density of particles. This technique was initiated in the seminal paper [von Smoluchowski, 1917] when the driving stochastic process is a Brownian motion and particle collisions result in perfect agglomeration. The author presented an explicit collision kernel and a coagulation equation (from which other population balance equation have been derived). Several extensions are presented in [Friedlander, 1977] by introducing a drag force on the particles. In [Meyer and Deglon, 2011] there is a review of many different kernel expressions that have been proposed but only in the case of inertial particles.

However, it is not straightforward to generalise these expressions of collision kernels to more complex situations such as partial absorption, reflection, interaction terms between the particles. One technique to obtain collision kernel models is through experimentation, but it can be difficult to extract the data, especially when particles are very small. Also it can be quite a costly procedure.

Another technique is based on the second approach mentioned earlier: particle tracking and numerical simulation. In this thesis we shall consider and analyse models inspired by this latter approach, by proposing a discretisation scheme for the particle dynamics and an efficient algorithm to simulate the scheme.

As mentioned, for particle tracking simulation, one must select a model for the movement of the particles and the type of interaction between them.

Dynamics of particles

There are two main classes of stochastic models for small (colloidal) particle dynamics.

Historically, the first stochastic model for particle motion was proposed by [Einstein, 1905]. In his seminal article, a definition for the Brownian motion was introduced using arguments based on thermal molecular fluctuations. In the same article, the Fokker-Planck equation is presented with a diffusion coefficient derived from thermodynamic equilibrium considerations. Later in his Nobel recognised research, J.B. Perrin calculated an empirical value for the diffusion coefficient which matched Einstein's analytical value thus establishing the atomic nature of matter.

The second class for a stochastic model of the motion was given by [Langevin, 1908] who assumed colloidal particles followed a kinetic model with a drag component and a stochastic forcing on the velocity. By applying Newton's equation, we obtain an SDE and in the over-damped limit (as the drag force goes to infinity), it is possible to show that the Langevin model converges to the Einstein one, thus proving the consistency between the models.

From now on, we shall focus mainly on the class of kinetic, Langevin models. Thus the model we consider is the case of Langevin particles in turbulent flow which we can express in a generic manner as:

$$\begin{cases} dx_t = u_t\,dt \\ du_t = \text{Fluid Velocity Term}\,dt \underbrace{-\lambda u_t\,dt}_{\text{Drag Force}} + B\,dW_t \end{cases} \tag{0.1}$$

where B is a constant and $(W_t)_{t\geq 0}$ is a standard Brownian motion and λ is a drag coefficient. In [Minier and Peirano, 2001], we have a specific example of such a process in the case of low Reynolds number. The Fluid Velocity Term is plugged in from external sources such as a Direct Numerical Simulation, thus an important assumption we make is that the particles do not influence the fluid velocity.

Interaction between particles: perfectly elastic collision

After selecting a model for the dynamics, we now consider the interaction between particles. We shall assume that the interactions happen when two particles come into contact (collision).

In the framework of such Langevin models, the analysis of the collision kernel is further developed by considering the case when collisions are followed by perfectly elastic reflections. This particle tracking will later be used in more complicated situations, such as turbulent flows, where closed-form expressions for the collision kernel do not exist. The main difference between the Brownian and Langevin models is that the set of crossings of a level by a Brownian particle is uncountable while for a Langevin particle it is separated as seen in [McKean, 1962]. Thus in a Langevin situation it is much more obvious to define a collision rate as the collisions can simply be counted.

In order to simplify our model, we shall consider the case of two particle collisions between colloids of same dimensions and mass. Perfectly elastic reflections are defined as collisions where the linear momentum and kinetic energy are conserved which, in our case of identical particles, involve reversing the velocity component that is normal to the collision plane. This can be seen in the Figure 1 where we present a collision that takes place at time t_c. The normal to the collision plane component of the velocity of particle p_1, in red, is transferred to the incoming particle p_2, during the collision, at t_c and vice-versa.

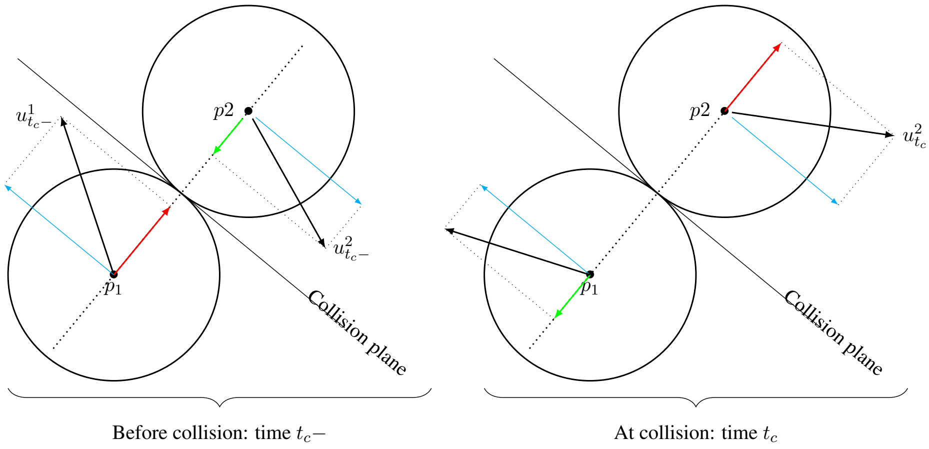


Figure 1: Perfectly elastic reflection

Another simplification will be performed by taking the relative position and velocity between the particles. This will eliminate the need to track both particles involved in a collision. Under such a transformation, the perfectly elastic collision becomes a specular reflection condition for the relative process. Specular reflection is just perfectly elastic reflection against a fixed wall. In Figure 2, we plot the perfectly elastic collision between two mono-dimensional particles in the reference frame of particle p_1. The coloured arrows represent the velocities of the particles in a fixed reference where the reflection would appear, while the dark arrows represent the relative position and velocity with respect to particle p_1. We can notice that after the reflection at time t_c, the value of the relative velocity is the opposite of the relative velocity just before the collision at t_c-.

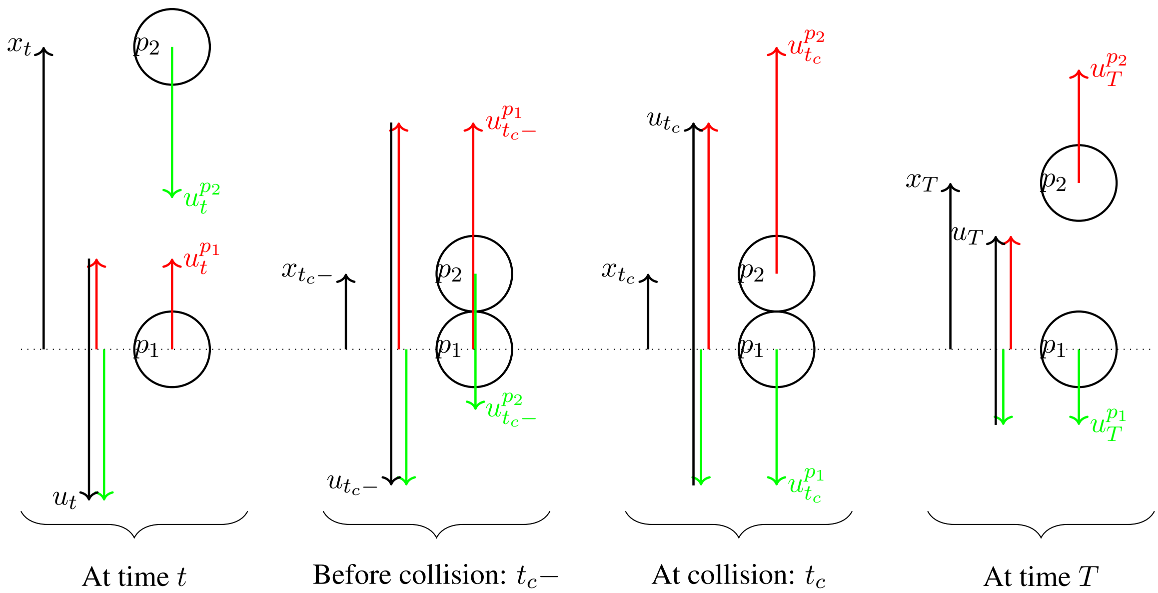


Figure 2: Perfectly elastic reflection between two particles and relative distance and velocity

In a one-dimensional setting, the generic process (0.1) is transformed as:

$$\begin{cases} dx_t = u_t\,dt \\ du_t = \text{Fluid Velocity Term}\,dt - \lambda u_t + B\,dW_t + d(\text{Jump Term}) \end{cases} \tag{0.2}$$

where Jump Term includes the fact that at the moment of collision the relative velocity is reversed, thus a time-discontinuous term is needed.

We can mention that in the case of reflections against a fixed wall, $(x_t, u_t)_{t\geq 0}$ represent the actual position and velocity of the particle and not relative quantities.

Models that include a specular type condition have already been introduced in [Dreeben and Pope, 1997] which uses such a condition to impose a wall boundary condition on fluid particles in the logarithmic layer. In [Minier and Pozorski, 1999], the authors also use a specular reflection condition in an alternative approach, when looking for the PDF model equivalent of wall functions.

In order to simulate the collision kernel in more complex engineering situations, one needs first to understand the simulation of two particle collisions which can be reduced to specular reflection. The thesis also considers that the particle dynamics are modelled by a Langevin process. It presents theoretical and numerical results for the simulation of a proposed discretisation scheme. A general objective was to propose a scheme that can be implemented for a large number of particles. Our suggested algorithm of simulation can be used in such a context since the method of intersecting collision cylinders would still apply (i.e. the motion of particles through space and time describes a cylinder, and if two cylinders intersect, that means a collision took place).

1 Overview of the thesis

The thesis is structured in 3 chapters. In the first chapter, we analyse a simulation scheme for the Langevin model with specular reflection that is similar to the one proposed in [Bernardin *et al.*, 2010] which used the stochastic Lagrangian approach for fluid particles. The error of the scheme is considered in a weak sense, meaning that only approximations of any statistic on the particles in position and velocity, at fixed time, are taken into account. An order of convergence for the weak error under certain hypotheses is presented. In the second chapter, we put forward a numerical validation of the proven theoretical results. Also numerical results from penalised schemes are presented for comparison purposes with our proposed scheme. Finally, we recall that a free Langevin process will converge in a certain sense to a Brownian process, so in the final chapter we shall present some non-asymptotic bounds on the weak error between the Langevin process with specular reflection and the reflected Brownian process.

Structure and contents of the first chapter: The Symmetrised Scheme for the Stochastic Lagrangian Model with Specular Reflection

In the first chapter we introduce the discretisation scheme and offer a theoretical order of convergence for the weak error. To do this, the classical method that involves the regularity of the solutions to the Kolmogorov problem is used. Thus, in order to prove:

- *first order regularity* we used the connection between reflected Langevin model and non-confined Langevin model with a modified drift to analyse the regularity of the flow. This required results from [Bouleau and Hirsch, 1989]
- *higher order regularity* we have utilised energy inequalities obtained from the variational formulation of the Kolmogorov problem.

In the Appendix to this chapter, a slight extension to results from [Bossy and Jabir, 2015] is made to prove the well-posedness of the Feynman-Kac formula when adding a drift term.

Structure of the second chapter: Empirical Analysis Based on Numerical Experiments

The theoretical results proven in the first chapter are tested in a numerical simulation setting.

- Primary objective : test the theoretical rate for the weak error obtained in the previous chapter, on a panel of test cases (1) explicit solution, (2) hypotheses satisfied for evaluation function and drift (3) hypotheses not satisfied for the drift.
- Establish comparisons with other proposed schemes in the literature for the weak error
- As schemes are proposed for strong convergence, examine the strong convergence rate.
- Then we use the strong convergence rate to propose a multi-level MC procedure and combine this to go back to the primary objective.

Structure of the third chapter: Non-asymptotic Approximations of the Langevin Equation by a Diffusion in the case of particle collision

This chapter contains calculations on the error rate produced when approximating the stiff Langevin (fast-slow) equation by a certain diffusion process with drift and diffusion coefficients that depend on those of the Langevin.

- Calculations are shown in the case of a driftless system.
- The mild equation is utilised to obtain similar results in the case of a drift.
- Similar calculations are performed in the case of reflection models.

Chapter 1

The Symmetrised Scheme for the Stochastic Lagrangian Model with Specular Reflection

1 Introduction

Many industrial production processes involve suspensions of colloidal particles in fluids so there is a strong interest to better understand the underlying physics. Among the ways that can help to achieve this goal, numerical experiments combining the simulation of the flow and the simulation of the particles carried by the flow is a possible solution. Propositions of model-motion of colloidal particles are already well-known, assuming that they can be modelled by small spheres and that the description of the model motions of their gravity centres is a significant approximation when one want to asses some characteristic behaviour through collision kernel modelling.

In this chapter, we propose and analyse the convergence of a time-discretisation scheme for the motion of a particle when the instantaneous velocity of the particle is drifted by the known velocity of the carrying flow, and when the motion is taking into account the collision event with a boundary wall.

More precisely, since we want to work in a context where we can specify the mathematical well-posedness of the problem and regularity for the solutions of associated PDEs, some simplifications are considered. We assume that the collision is perfectly elastic and that the particles follow a kinetic model, by modeling the position and velocity of each particle. It is on the velocity that we introduce a drift term to model the influence of the fluid on the particles. Furthermore, we will only consider a particle that collides against a wall located at the boundary of the upper-half plane $\mathbb{R}^{d-1} \times [0, +\infty)$. In this case the confined linear Langevin process is written as:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s \, ds, \\ U_t = U_0 + \displaystyle\int_0^t b(X_s, U_s) \, ds + \sigma W_t + K_t, \\ K_t = - \displaystyle\sum_{0<s\leq t} 2 \left(U_{s^-} \cdot n_{\mathcal{D}}(X_s) \right) n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \end{cases} \tag{1.1}$$

where $(X_t)_{t\geq 0}$ represents the position while $(U_t)_{t\geq 0}$ represents the velocity, $\mathcal{D} := \mathbb{R}^{d-1} \times (0, +\infty)$ is the open set corresponding to the interior of the confining domain, $n_{\mathcal{D}}$ is the outward normal at the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ ($\partial\mathcal{D} = \mathbb{R}^{d-1} \times \{0\}$) and σ is a positive constant. Here the drift b models the drag force implied by the known mean velocity of the flow carrying the particle. The term $(K_t)_{t\geq 0}$ represents the perfectly elastic collision with the hyperplane $\partial\mathcal{D}$.

Although simple -known as specular reflection against a fixed wall- this model contains enough characteristics of the context stated in the first paragraph to be pertinent on a framework of numerical analysis. In [Bossy and Jabir, 2011], Bossy and Jabir prove the existence of weak solution and pathwise uniqueness when $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$. In [Bossy and Jabir, 2015], the authors extend the well-posedness result to smooth bounded domains $\mathcal{D}$. In the case of hyperplane $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$, the construction proceeds as follows (see [Bossy and Jabir, 2011] for the details). If we consider a $\mathbb{R}^d$-valued bounded measurable drift $\tilde{b}$ on $\mathcal{D} \times \mathbb{R}^d$, from the unique weak $\mathbb{R}^{2d}$-valued solution of

$$\begin{cases} Y_t = X_0 + \int_0^t V_s ds, \\ V_t = U_0 + \int_0^t \tilde{b}(Y_s, V_s) ds + \widetilde{W}_t, \ \forall\, t \in [0, T], \end{cases} \tag{1.2}$$

with $\tilde{b}$ defined by

$$\tilde{b}\colon (y, v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(b', \operatorname{sign}(y^{(d)}) b^{(d)}\right)\left((y', |y^{(d)}|), (v', \operatorname{sign}(y^{(d)}) v^{(d)})\right) \tag{1.3}$$

then

$$((Y_t', |Y_t^{(d)}|), (V_t', \operatorname{sign}(Y_{t+}^{(d)}) V_t^{(d)}); t \in [0, T])$$

is the weak solution in $\mathcal{D} \times \mathbb{R}^d$ to the SDE (1.1).

A short discussion on confined SDEs and associated results

There are different types of confined models that can be considered. In a deterministic setting, [Paoli and Schatzman, 1993] present some results when $\sigma = 0$ in (1.1) while allowing for oblique reflections. The authors show that the system admits a solution such that the position process is Lipschitz continuous in time, and the velocity process is of bounded variation. This solution is obtained as a certain weak limit in a Sobolev space of solutions to a penalized equation.

The most obvious stochastic model would be a diffusion that is reflected at the boundary, in the sense of a solution to a Skorohod problem as in [Lions and Sznitman, 1984]. The reflection term K is then given through a local time. In term of discretisation scheme, [Bossy *et al.*, 2004], propose a symmetrized scheme, and prove that the associated weak error has a rate of convergence of order one.

In [Costantini, 1991], the author presents a model with a particle that exhibits piecewise deterministic movement. The velocity process changes randomly at exponential times to mimic the collision events. The particles are confined in domain by specular reflections at the boundary. It is shown that such a system is well defined and by increasing the change rate for the velocity, in the limit, one obtains an oblique reflected diffusion.

We emphasize the fact that, when modelling the position of the particle by a reflected Brownian process, the hitting times of the boundary form almost surely a set of times with no isolated points. This means that it is impossible to count the number of collisions with the boundary. Those models are not suitable in numerical approach when one might to determine a collision kernel with the help of the effective collision rate. Such inconvenient disappears by considering models for the particle collisions of Lagrangian type, where the position process is the integral of a diffusion. As shown in [McKean, 1962], situation of accumulation of collisions can be avoided for Lagrangian models in the case of a upper half plane under the hypothesis that $(X_0, U_0) \neq (0, 0)$.

We also mention that the case of absorbing boundary have been studied in [Bertoin, 2007] and in [Jacob, 2012], [Jacob, 2013] who prove the existence of a reflecting Langevin process with an absorbing boundary.

Finally, in [Costantini, 1992] and in [Spiliopoulos, 2007], it have been shown that using a certain type of scaling and limit in the drift and diffusion parameters in (1.1), it is possible to pass from a Langevin model with specular reflection (1.1) to a reflected diffusion model for the position process.

Discretization scheme for the confined SDE (1.1)

Without any loss to the generality, we present a discretisation scheme in case of the dimension $d = 1$. The scheme can be easily generalized to higher dimensions by combining the discretisation of the first $d-1$ components of the process $(X_t, U_t)_{t\geq 0}$, solution of (1.1), using standard discretisation scheme in $\mathbb{R}^{d-1}$, and the confined scheme presented in this section for the dth component.

As previously mentioned, in [Bossy and Jabir, 2011] the authors construct a weak solution to the equation (1.1) when the reflection border is a hyperplane. The position process of this weak solution is written as the absolute value of an unconfined Langevin process. The following scheme borrows the main ideas of this transformation by symmetry.

The confined process is discretised on an a regular mesh $0 = t_0 < t_1 < \ldots < t_n = T$ of the interval $[0, T]$. $\Delta t = t_{i+1} - t_i$ is the time increment. We define the discretised process $(\bar{X}_t, \bar{U}_t)_{0\leq t\leq T}$ with an iterative procedure. Knowing (X_{t_i}, U_{t_i}) we construct $(X_{t_{i+1}}, U_{t_{i+1}})$ as follows:

• **Discretisation of the position process.** We denote by $(\bar{Y}_t)_{0\leq t\leq T}$ the prediction step of a new position. The approximation process $(\bar{X}_t)_{0\leq t\leq T}$ is simply obtained from $(\bar{Y}_t)$ by taking the absolute value of the prediction :

$$\begin{cases} \bar{Y}_{t_{i+1}} = \bar{X}_{t_i} + (t_{i+1} - t_i)\bar{U}_{t_i} \\ \bar{X}_{t_{i+1}} = |\bar{Y}_{t_{i+1}}|. \end{cases} \tag{1.4}$$

A collision of the discretised particle with the wall boundary takes place during the time interval $(t_i, t_{i+1}]$, if $t_i < t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}} \leq t_{i+1}$. We introduce the sequence of times $(\theta_i, i = 1, \ldots, n)$ defined as

$$\theta_i = \begin{cases} t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}}, & \text{if } t_i < t_i - \dfrac{\bar{X}_{t_i}}{\bar{U}_{t_i}} \leq t_{i+1}, \\ t_i, & \text{otherwise.} \end{cases} \tag{1.5}$$

We call the (θ_i) the collision times (expect when $\theta_i = t_i$), and we observe that when $\theta_i > t_i$,

$$\bar{Y}_{\theta_i} = \bar{X}_{\theta_i} = 0.$$

• **Discretisation of the velocity process.**

$$\begin{cases} \text{if } \theta_i \in (t_i, t_{i+1}], \text{ a collision takes place during the interval:} \\ \quad \text{for } t_i \leq t < \theta_i \\ \quad\quad \bar{U}_t = \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(t - t_i) + \sigma(W_t - W_{t_i}) \\ \quad \text{at } \theta_i, \text{ velocity reflection :} \\ \quad\quad \bar{U}_{\theta_i} = -\bar{U}_{\theta_i^-} \\ \quad \text{for } \theta_i < t \leq t_{i+1} : \\ \quad\quad \bar{U}_t = \bar{U}_{\theta_i} + b(\bar{X}_{\theta_i}, \bar{U}_{\theta_i})(t - \theta_i) + \sigma(W_t - W_{\theta_i}) \\ \text{else, no collision :} \\ \quad \text{for } t_i \leq t \leq t_{i+1} \\ \quad\quad \bar{U}_t = \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(t - t_i) + \sigma(W_t - W_{t_i}). \end{cases} \tag{1.6}$$

When $d > 1$, the scheme writes exactly the same, except that one have to adapt the computation of the collision time and the velocity reflection as

$$\theta_i = t_i + \frac{\bar{X}^{(d)}_{t_i}}{(\bar{U}_{t_i} \cdot n_{\mathcal{D}})}$$

and

$$(\bar{U}_{\theta_i} \cdot n_{\mathcal{D}}) = -(\bar{U}_{\theta_i^-} \cdot n_{\mathcal{D}})).$$

Similar schemes to the one presented above have been applied for confined and McKean non linear Lagrangian models involved in the modelling of turbulent atmospheric flow (see [Bernardin *et al.*, 2010] and [Bossy *et al.*, 2016]). In particular, particles collisions with the boundary simulation domain are used to impose Dirichlet boundary condition for the velocity. The scheme is also implemented in the WindPos[1] software for wind simulation and wind farms based on fluid particle simulation.

In what follows, we prove the first rate of convergence result for the weak error produce by such scheme.

1.1 Main result

Let us first introduce hereafter our hypotheses. From now on, we implicitly assume that σ is strictly positive. A first set of hypotheses ($H_{Langevin}$) is needed to insure the existence of a solution to the system (1.1). A second set (H_{PDE}) insures the existence and the regularity of a solution to the backward Kolmogorov PDE associated to the SDE (1.1). A third set ($H_{Weak\ Error}$) is added to insure the weak convergence rate of order one.

Hypotheses 1.1

($H_{Langevin}$)-(i) The initial condition (X_0, U_0) is assumed to be distributed according to a given initial law μ_0 having its support in $\mathcal{D} \times \mathbb{R}^d$ and such that $\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x|^2 + |u|^2\right) \mu_0(dx, du) < +\infty$.

($H_{Langevin}$)-(ii) The drift $b\colon \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is uniformly bounded and Lipschitz-continuous with constant $\|b\|_{Lip}$.

(H_{PDE})-(i) The drift b is a $\mathcal{C}_b^{1,1}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ function, and the first derivatives $\nabla_x b$ and $\nabla_u b$ are also Lipschitz on $\mathbb{R}^d \times \mathbb{R}^d$.

(H_{PDE})-(ii) When $x \in \partial\mathcal{D}$, the d^{th} coordinate of $u \mapsto b(x,u)$ is an odd function in terms of the d^{th} coordinate of the variable u. The first $(d-1)$ coordinates of $b(x,u)$ (denoted $b'(x,u)$) are even functions with respect to the same d^{th} coordinate of the variable u. In particular, for any $x = (x', 0) \in \partial\mathcal{D}$ and $u \in \mathbb{R}^d$,

$$b(x,u) = (b', b^{(d)})((x',0),(u',u^{(d)})) = (b', -b^{(d)})((x',0),(u',-u^{(d)})),$$

where for any vector $v \in \mathbb{R}^d$, v' denotes the $d-1$ firsts components and $v^{(d)}$ denotes the d^{th} one.

($H_{Weak\ Error}$)-(i) μ_0 admits a Lebesgue density function that is still denoted μ_0 in $L^\infty(\mathcal{D} \times \mathbb{R}^d)$ and there exists $\varepsilon_0 > 0$ such that

$$\frac{\inf\{x; (x,u) \in \mathrm{Supp}(\mu_0)\}}{\inf\{u; (x,u) \in \mathrm{Supp}(\mu_0) \text{ and } u < 0\}} < -\varepsilon_0.$$

Remark 1.2. *The results presented below remain valid if we assume that the drift b is also time dependent with $b \in \mathcal{C}^1((0,T); \mathcal{C}_b^{1,1}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d))$ and $\nabla_x b$, $\nabla_u b$ are Lipschitz.*

Remark 1.3. *The condition (H_{PDE})-(ii) restricts strongly the set of drifts b for which we can claim a first order convergence rate for the weak error. However a typical example of drift b, coming from the application of colloidal particles carrying by a flow, respects this condition. A particle in a flow undergo a drag force that is modeled in the velocity equation as*

$$b(t,x,u) = -k(t,x)(u - \mathcal{V}(t,x)),$$

[1] see `https://windpos.inria.fr`

where $\mathcal{V}(t,x)$ is the velocity of the fluid seen by the particle at position x and at the time t. In a laminar or turbulent flow, a no-permeability condition at the wall is imposed, that implies that for all $x \in \partial\mathcal{D}$,

$$(\mathcal{V}(t,x) \cdot n_{\mathcal{D}}(x)) = 0.$$

In our case of hyperplane $\mathcal{D}$, this means that for $(x,u) \in \partial\mathcal{D} \times \mathbb{R}$, $\mathcal{V}^{(d)}(t,x) = 0$ and

$$b^{(d)}(x,u) = b^{(d)}(x,u^{(d)}) = -k(t,x)\,u^{(d)}.$$

For such important example, for $x \in \partial\mathcal{D}$, $b^{(d)}(x,\cdot)$ is odd in $u^{(d)}$ and the b' components do not depend on $u^{(d)}$ and satisfy (H_{PDE})-(ii).

Remark 1.4. *Later in the proofs, we will introduce again the transformed drift $\widetilde{b}$ used in (1.3) to construct a solution to (1.1) and defined as*

$$\tilde{b}\colon (y,v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \Big(b', \operatorname{sign}(y^{(d)})b^{(d)}\Big)\big((y', |y^{(d)}|), (v', \operatorname{sign}(y^{(d)})v^{(d)})\big).$$

where the function sign *is defined in (1.12). Hypotheses ($H_{Langevin}$)-(ii) and (H_{PDE})-(ii) ensure the continuity of $\widetilde{b}$. Indeed, for $(y,v) \in (\mathbb{R}^d \setminus \partial\mathcal{D}) \times \mathbb{R}^d$, by the hypothesis ($H_{PDE}$)-($i$), we have that $\widetilde{b}$ is continuous at (y,v).*

Let $(y,v) \in \partial\mathcal{D} \times \mathbb{R}^d$, then by the evenness condition in (H_{PDE})-(ii), we have that

$$\widetilde{b}'(y,v) = b'((y',0),(v',-v^{(d)})) = b'((y',0),(v',v^{(d)})) = \lim_{h\searrow 0} b((y',h),v) = \lim_{h\searrow 0} \widetilde{b}'((y',h),v).$$

and

$$\widetilde{b}^{(d)}(y,v) = -b^{(d)}((y',0),(v',-v^{(d)})) = b^{(d)}((y',0),(v',v^{(d)})) = \lim_{h\to 0} \widetilde{b}^{(d)}((y',h),v).$$

By ($H_{Langevin}$)-(ii), $\widetilde{b}$ is also piecewise Lipschitz. Together with the continuity property, $\widetilde{b}$ is uniformly Lipschitz with a Lipschitz constant $\|\tilde{b}\|_{Lip}$ equal to $2\|b\|_{Lip}$.

Indeed, for $i = 1,\ldots,d-1$,

$$\begin{aligned}
&|\tilde{b}^{(i)}(x,u) - \tilde{b}^{(i)}(y,v)| \\
&= |(b')^{(i)}\big((x',|x^{(d)}|),(u',\operatorname{sign}(x^{(d)})u^{(d)})\big) - (b')^{(i)}\big((y',|y^{(d)}|),(v',\operatorname{sign}(y^{(d)})v^{(d)})\big)| \\
&\leq \mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=1\}}\left\{\|b\|_{\mathrm{Lip}}\left(\|x-y\|+\|u-v\|\right)\right\} \\
&\quad + \mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=-1\}}\left|(b')^{(i)}\big((y',|y^{(d)}|),(v',\operatorname{sign}(y^{(d)})v^{(d)})\big) - (b')^{(i)}\big(0,(v',\operatorname{sign}(y^{(d)})v^{(d)})\big)\right| \\
&\quad + \mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=-1\}}\left|(b')^{(i)}\big(0,(v',\operatorname{sign}(y^{(d)})v^{(d)})\big) - (b')^{(i)}\big(0,(u',\operatorname{sign}(x^{(d)})u^{(d)})\big)\right| \\
&\quad + \mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=-1\}}\left|(b')^{(i)}\big((x',|x^{(d)}|),(u',\operatorname{sign}(x^{(d)})u^{(d)})\big) - (b')^{(i)}\big(0,(u',\operatorname{sign}(x^{(d)})u^{(d)})\big)\right|.
\end{aligned}$$

Using hypothesis (H_{PDE})-(ii), the third term above is bounded by $\|b^{(i)}\|_{\mathrm{Lip}}\|u-v\|$. Moreover, since $\mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=-1\}}(\|x\|+\|y\|) \leq 2\|x-y\|$, we conclude that $|\tilde{b}^{(i)}(x,u)-\tilde{b}^{(i)}(y,v)| \leq 2\|b^{(i)}\|_{\mathrm{Lip}}(\|x-y\|+\|u-v\|)$. Similarly, for the d-component, using several times that for any (x,y,u,v),

$$\mathbb{1}_{\{\operatorname{sign}(x^{(d)}y^{(d)})=-1\}}\left(\operatorname{sign}(y^{(d)})b^{(d)}\big(0,(v',\operatorname{sign}(y^{(d)})v^{(d)})\big) - \operatorname{sign}(x^{(d)})b^{(d)}\big(0,(v',\operatorname{sign}(x^{(d)})v^{(d)})\big)\right) = 0,$$

we obtain with the same decomposition that as well that,

$$|\tilde{b}^{(d)}(x,u) - \tilde{b}^{(d)}(y,v)| \leq 2\|b^{(d)}\|_{\mathrm{Lip}}(\|x-y\|+\|u-v\|).$$

Remark 1.5. *The condition ($H_{Weak\ Error}$)-(i) on the support of μ_0 implies that the first collision time of the scheme (1.4)-(1.6) is almost surely separated from $t=0$. In the proposed scheme, the first possible collision time before Δt is*

$$-\frac{X_0}{U_0} \geq \varepsilon_0 > 0.$$

Rate of convergence result

We denote by Q_T the set $(0,T) \times \mathcal{D} \times \mathbb{R}^d$. For any measurable function ψ defined on $\overline{\mathcal{D}} \times \mathbb{R}^d$, we consider the function $F : Q_T \longrightarrow \mathbb{R}$ defined as

$$F(t,x,u) = \mathbb{E}\psi(X_T^{t,x,u}, U_T^{t,x,u}) \tag{1.7}$$

where the process $(X_s^{t,x,u}, U_s^{t,x,u})_{s\geq t}$ solves the SDE (1.1) that begins at time t with values (x,u).

Our main result is the following

Theorem 1.6. *Assume ($H_{Langevin}$), (H_{PDE}) and ($H_{Weak\ Error}$) and fix $T > 0$. Then, for any test function $\psi \in \mathcal{C}_c^{1,1}(\mathcal{D}, \mathbb{R}^d; \mathbb{R})$, there exists a constant C_{F,σ,b,T,μ_0} such that we can prove a first order convergence bound for the weak approximation error:*

$$\left|\mathbb{E}\psi(X_T, U_T) - \mathbb{E}\psi(\bar{X}_T, \bar{U}_T)\right| \leq C_{F,\sigma,b,T,\mu_0}\, \Delta t \tag{1.8}$$

where C_{F,σ,b,T,μ_0} depends only on the solution F to the PDE (1.9) and their derivatives, on the drift b and their derivatives, on the diffusion constant σ, on the terminal time T and on the norm $\|\mu_0\|_{L^\infty}$ of the initial density distribution of (X_0, U_0).

A key argument in the proof of the theorem resides in the regularity we can show for the function F.

We start, showing first in section 6 that when ψ is in $\mathcal{C}_c(\mathcal{D}, \mathbb{R})$, F is a weak solution to the following backward Kolmogorov PDE (see Proposition 6.4) with specular boundary condition:

$$\begin{cases} \partial_t F + (u\cdot\nabla_x F) + (b(x,u)\cdot\nabla_u F) + \dfrac{\sigma^2}{2}\triangle_u F = 0, \text{ on } Q_T, \\ F(T,x,u) = \psi(x,u), \text{ on } \mathcal{D}\times\mathbb{R}^d, \\ F(t,x,u) = F(t,x,u - 2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+. \end{cases} \tag{1.9}$$

with Q_T defined at (1.10) and Σ_T^+ defined in (1.11). A priori L^2 bound for first order derivatives of F is shown in Section 4. The proof of this result is based on the probabilistic expression of F in (1.7). Section 5 is dedicated to higher order regularity result using L^2 energy inequality formulation. Furthermore, in section 4 we show that the first derivatives are in $L^\infty(Q_t)$.

Section 2 presents a schematic proof of the weak error rate in the case of a diffusion without any boundaries. We also introduce some results needed for the proof of the main theorem. The proof of Theorem 1.6 is given in section 3 and is based on regularity obtained on F.

In order to simplify notations, the analysis for Section 3 is given assuming $d = 1$. In the other sections, the dimension d is arbitrary, unless it is explicitly mentioned.

1.2 Notation

The space $\mathcal{C}_b^{l,m}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ is the set of continuous and bounded functions on $\mathbb{R}^d \times \mathbb{R}^d$, with continuous and bounded derivatives with respect to the variables in $\mathbb{R}^d \times \mathbb{R}^d$, up to the order l and m respectively.

The space $\mathcal{C}_c^{l,m}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ has the same definition but for functions with compact supports.

The space $\mathcal{C}_c^{l}(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$ with compact supports, with continuous and bounded derivatives up to the order l.

For all $t \in (0,T]$, we introduce the time-phase space

$$Q_t := (0,t) \times \mathcal{D} \times \mathbb{R}^d, \tag{1.10}$$

the outward normal to $\mathcal{D}$ noted by $n_{\mathcal{D}}$ and the boundary sets:

$$\begin{array}{ll} \Sigma^{+} := \left\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) > 0\right\}, & \Sigma_t^{+} := (0,t) \times \Sigma^{+}, \\ \Sigma^{-} := \left\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) < 0\right\}, & \Sigma_t^{-} := (0,t) \times \Sigma^{-}, \\ \Sigma^{0} := \left\{(x,u) \in \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) = 0\right\}, & \Sigma_t^{0} := (0,t) \times \Sigma^{0}, \end{array} \tag{1.11}$$

and further $\Sigma_T := \Sigma_T^{+} \cup \Sigma_T^{0} \cup \Sigma_T^{-} = (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d$. Denoting by $d\sigma_{\partial\mathcal{D}}$ the surface measure on $\partial\mathcal{D}$, we introduce the product measure on Σ_T:

$$d\lambda_{\Sigma_T} := dt \otimes d\sigma_{\partial\mathcal{D}}(x) \otimes du.$$

We introduce the Sobolev space

$$\mathcal{H}(Q_t) = L^2((0,t) \times \mathcal{D}; H^1(\mathbb{R}^d))$$

equipped with the norm $\| \, \|_{\mathcal{H}(Q_t)}$ defined by

$$\|\phi\|^2_{\mathcal{H}(Q_t)} = \|\phi\|^2_{L^2(Q_t)} + \|\nabla_u \phi\|^2_{L^2(Q_t)}.$$

We denote by $\mathcal{H}'(Q_t)$, the dual space of $\mathcal{H}(Q_t)$, and by $(\; , \;)_{\mathcal{H}'(Q_t),\mathcal{H}(Q_t)}$, the inner product between $\mathcal{H}'(Q_t)$ and $\mathcal{H}(Q_t)$.

We further introduce the space

$$L^2(\Sigma_T^{\pm}) = \left\{\psi : \Sigma_T^{\pm} \to \mathbb{R} \text{ s.t } \int_{\Sigma_T^{\pm}} |(u \cdot n_{\mathcal{D}}(x))| \, |\psi(t,x,u)|^2 \, d\lambda_{\Sigma_T}(t,x,u) < +\infty\right\},$$

equipped with the norm

$$\|\psi\|_{L^2(\Sigma_T^{\pm})} = \sqrt{\int_{\Sigma_T^{\pm}} |(u \cdot n_{\mathcal{D}}(x))| \; |\psi(t,x,u)|^2 \, d\lambda_{\Sigma_T}(t,x,u)}.$$

The space $L^2(\Sigma_T)$ is defined, through the respective restriction on $\Sigma_T^{\pm}$ denoted $|_{\Sigma_T^{\pm}}$ as

$$L^2(\Sigma_T) = \left\{\psi : \Sigma_T \to \mathbb{R} \text{ s.t } \psi|_{\Sigma_T^{\pm}} \in L^2(\Sigma_T^{\pm})\right\},$$

and equipped with the norm

$$\|\psi\|_{L^2(\Sigma_T)} = \quad \|\psi|_{\Sigma_T^{+}}\|_{L^2(\Sigma_T^{+})} + \|\psi|_{\Sigma_T^{-}}\|_{L^2(\Sigma_T^{-})}.$$

The following convention for the function $\operatorname{sign}\colon x \in \mathbb{R} \mapsto \mathbb{R}$ is considered:

$$\operatorname{sign}(x) = \begin{cases} -1, & \text{for } \; x \leq 0 \\ \;\;\,1, & \text{for } \; x > 0 \end{cases} \tag{1.12}$$

For multidimensional functions, we use the following definition of L^2 space:

$$L^2(Q_T; \mathbb{R}^d) = \left\{\psi : Q_T \to \mathbb{R}^d \text{ s.t } \int_{Q_T} \|\psi\|^2 < +\infty\right\},$$

where $\|\|$ is the Euclidean norm on $\mathbb{R}^d$.

$$L^2(Q_T; \mathbb{R}^{d\times d}) = \left\{\psi : Q_T \to \mathbb{R}^{d\times d} \text{ s.t } \int_{Q_T} \|\psi\|_F^2 < +\infty\right\},$$

where $\|\|_F$ is the Frobenius norm i.e. for any $d\times d$ matrix A, $\|A\|_F = \sqrt{\mathrm{Tr}(AA^T)} = \sqrt{\sum_{i=1}^d \sum_{j=1}^d |a_{ij}|^2}$.

We denote by $\mathrm{Jac}_x(\psi) = \left(\dfrac{\partial \psi_i}{\partial x_j}\right)_{1\le i,j\le d}$ the Jacobian matrix of $\psi : \mathbb{R}^d \to \mathbb{R}^d$ w.r.t x and $\mathrm{Hess}_{x,u}(\varphi) = \left(\dfrac{\partial^2 \varphi}{\partial x_i \partial u_j}\right)_{1\le i,j\le d}$ is the Hessian matrix w.r.t (x,u) of $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

For any functions $G\colon Q_T \mapsto \mathbb{R}$, $\gamma_1\colon \mathbb{R} \mapsto \mathbb{R}$, $\gamma_2\colon \mathbb{R}^d \mapsto \mathbb{R}$ and $\gamma_3\colon \mathbb{R}^d \mapsto \mathbb{R}$, we define the joint convolution $G * (\gamma_1\gamma_2\gamma_3)$ at any $(t,x,u) \in \overline{Q_T}$ as :

$$G * (\gamma_1\gamma_2\gamma_3)(t,x,u) := \int_{Q_T} G(\tau,y,v)\gamma_1(t-\tau)\gamma_2(x-y)\gamma_3(u-v)\, d\tau dy dv\,.$$

In case of multi-dimensional functions, the convolution applies on each components.

We will denote by $\|f\|_{\mathrm{Lip}}$ the Lipschitz constant of a function f from $\mathbb{R}^d$ to $\mathbb{R}^d$, defined as the smaller constant C such that

$$\|f(u) - f(u')\| \le C\|u - u'\|.$$

For a mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,u) \to f(x,u) \in \mathbb{R}^d$, we denote by $\|f\|_{\infty_x,\mathrm{Lip}_u}$, Lipschitz constant of f with respect to u, uniformly on x, defined as

$$\|f\|_{\infty_x,\mathrm{Lip}_u} = \sup_{x\in\mathbb{R}^d} \|f(x,\cdot)\|_{\mathrm{Lip}}.$$

2 Preliminaries

We present a schematic of the usual method to obtain the weak error convergence rate. Let's consider a process $(Z_t)_{0\le t\le T}$, defined on $\mathbb{R}$, that is simple and unconfined SDE:

$$dZ_t = b(Z_t)\,dt + \sigma\,dW_t$$

where b is a sufficiently smooth bounded function. It is well known (see e.g [Friedman, 2012]) that, for any ψ in $C_b^2(\mathbb{R})$, there exists a classical solution $g \in \mathcal{C}_b^{1,2}((0,T)\times\mathbb{R})$ to the backward PDE:

$$\begin{cases} \dfrac{\partial g}{\partial t} + b(z)\dfrac{\partial g}{\partial z} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 g}{\partial z^2} = 0 \\ g(T,z) = \psi(z) \quad \forall z \in \mathbb{R}, \end{cases}$$

such that $g(t,z) = \mathbb{E}\psi(Z_T^{t,z})$, where $(Z_\theta^{t,z}, \theta > t)$ is the flow solution starting from the point $Z_t^{t,z} = z$. We denote by $\mathcal{L}$ the infinitesimal generator of the process $(Z_t)_{t\ge 0}$ defined for any $h \in \mathcal{C}^2$ by:

$$\mathcal{L}h(z) = b(z)\frac{\partial h}{\partial z}(z) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial z^2}(z).$$

We introduce a regular time grid $0 = t_0 < t_1 < \ldots < t_n = T$, and the corresponding times-freezing function $\eta\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as $\eta(t) = t_i$ when $t \in [t_i, t_{i+1})$. We consider the continuous version $(\bar{Z}_t)_{t\ge 0}$ of the Euler scheme applied to Z as:

$$\bar{Z}_t = Z_0 + \int_0^t b(\bar{Z}_{\eta(s)})\,ds + \sigma\,W_t.$$

Now for any $\bar{z} \in \mathbb{R}$, we consider also $\mathcal{L}^{\bar{z}}$ the differential operator defined also on $\mathcal{C}^2$ functions by:

$$\mathcal{L}^{\bar{z}}h(z) = b(\bar{z})\frac{\partial h}{\partial z}(z) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial z^2}(z).$$

The weak error produced by the Euler scheme for the test function ψ can be obtained by applying the Itô's formula two successive times. From a first application, we get for a fixed $z \in \mathbb{R}$, since $g(0,z) = \mathbb{E}\psi(Z_T^{0,z})$,

$$\begin{aligned}
\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) &= \mathbb{E}\left[g(T,\bar{Z}_T^{0,z}) - g(0,z)\right] \\
&= \mathbb{E}\int_0^T \left(\partial_t g(t,\bar{Z}_t) + \mathcal{L}^{\bar{Z}_{\eta(t)}} g(t,\bar{Z}_t)\right) dt =
\end{aligned}$$

Since $\partial_t g + \mathcal{L}g = 0$, the previous equality becomes

$$\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) = \mathbb{E}\int_0^T \left(\mathcal{L}^{\bar{Z}_{\eta(t)}} g(t,\bar{Z}_t) - \mathcal{L}g(t,\bar{Z}_t)\right) dt = \mathbb{E}\int_0^T \frac{\partial g}{\partial z}(t,\bar{Z}_t)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_t)\right) dt$$

Now observing that for every time step t_i, we have that $\mathcal{L}^{\bar{Z}_{t_i}} g(t_i,\bar{Z}_{t_i}) = \mathcal{L}g(t_i,\bar{Z}_{t_i})$, by applying the Itô's formula once more on the interval $[\eta(t),t)$, we get

$$\begin{aligned}
\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z}) &= \mathbb{E}\int_0^T dt \int_{\eta(t)}^t ds \left(\mathcal{L}^{\bar{Z}_{\eta(t)}}\left(\frac{\partial g}{\partial z}(s,\bar{Z}_s)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_s)\right)\right)\right) \\
&\quad + \mathbb{E}\int_0^T dt \int_{\eta(t)}^t ds \left(\frac{\partial}{\partial s}\frac{\partial g}{\partial z}(s,\bar{Z}_s)\left(b(\bar{Z}_{\eta(t)}) - b(\bar{Z}_s)\right)\right).
\end{aligned}$$

Since g has bounded derivatives, the stochastic integrals from the applications of the Itô's formula are martingales.

The Δt factor, for the weak error convergence, is then extracted from the inner integral, since for any $t \in [0,T]$, $|t - \eta(t)| \le \Delta t$. If b is in $C_b^2(\mathbb{R})$ then there exists a constant K_T which depends on T such that for all $n = 0,1,2$, $|\partial_z^n g(t,z)| < K_T \|\psi\|_{W^{3,\infty}}$. This can be proven directly from $g(t,z) = \mathbb{E}\psi(Z_T^{t,z})$. Then, the previous equality can be bounded by

$$\left|\mathbb{E}\psi(\bar{Z}_T^{0,z}) - \mathbb{E}\psi(Z_T^{0,z})\right| \le C_{\partial^\alpha\varphi,\partial^\alpha b,\sigma,T}\Delta t$$

where $C_{\partial^\alpha\varphi,\partial^\beta b,\sigma,T}$ depends only on bounds for the derivatives of ψ up to the order 3, derivatives of b up to the order 2.

The proof of Theorem 1.6 is build on the same arguments, with certain particular differences that need to be adapted suitably:

- In Section 5, we prove that the solution to the Kolmogorov PDE (1.9) has some regularity in the $L^2(Q_T)$ space (see Theorem 2.1), instead of in $L^\infty(Q_T)$ space as in the previous sketch. Therefore the distribution of the initial values will be used to make appear L^2 norms in the previous arguments.

- Also, since we are interested in a confined SDE and backward PDE with specular condition, we will have to take into consideration boundary effects and adapt the form of the continuous version of the time discretization scheme.

- In order to apply Ito's formula as previously used, a time-continuous version of the schemes (1.4) and (1.6) need to be introduced. For this we consider first the function $\eta\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as previously as

  $$\eta(t) = t_i, \quad \forall t \in [t_i, t_{i+1}).$$

  Second, recalling the definition of the collision times in (1.5), we introduce $\nu\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined as:

  $$\nu(t) = \begin{cases} t_i & \text{for } t_i \le t < \theta_i \\ \theta_i & \text{for } \theta_i \le t < t_{i+1}. \end{cases} \tag{2.1}$$

We recall that θ_i is meant to signal if a collision is to take place on the interval $[t_i, t_{i+1})$. If there is a collision on this interval, then ν is t_i before the collision and θ_i after. If no collision takes place then ν is t_i.

With the help of $t \mapsto \eta(t)$ and $t \mapsto \nu(t)$, we write the continuous version of the discrete process as:

$$\begin{cases} \bar{Y}_t = \bar{X}_{\eta(t)} + (t - \eta(t))\bar{U}_{\eta(t)} \\ \bar{X}_t = X_0 + \displaystyle\int_0^t \bar{U}_{\eta(s)} \operatorname{sign}(\bar{Y}_s)\, ds \\ \bar{U}_t = U_0 + \displaystyle\int_0^t b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\, ds + \sigma W_t - 2 \sum_{0<s\leq t} \bar{U}_{s^-} \mathbb{1}_{\bar{X}_s = 0}. \end{cases} \tag{2.2}$$

2.1 The backward Kolmogorov PDE

We give some regularity results on the solution of the PDE (1.9).

Theorem 2.1. *Assume (H_{PDE}). When ψ belongs in $\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d;\mathbb{R})$, F defined in (1.7) belongs in $\mathcal{H}(Q_T)$, and is solution in the sense of distribution to the backward PDE:*

$$\begin{cases} \partial_t F + (u \cdot \nabla_x F) + (b(x,u) \cdot \nabla_u F) + \dfrac{\sigma^2}{2} \triangle_u F = 0, \; \text{on } Q_T, \\ F(T,x,u) = \psi(x,u), \; \text{on } \mathcal{D} \times \mathbb{R}^d, \\ F(t,x,u) = F(t,x,u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \; \text{on } \Sigma_T^+. \end{cases} \tag{1.9 bis}$$

When $\psi \in \mathcal{C}_c^{1,1}(\mathcal{D} \times \mathbb{R}^d;\mathbb{R})$, then F is in $\mathcal{C}([0,T]; L^\infty(\mathcal{D} \times \mathbb{R}^d);\mathbb{R}^d) \cap \mathcal{C}([0,T] \times \overline{\mathcal{D}} \times \mathbb{R}^d;\mathbb{R}) \cap L^2(Q_T;\mathbb{R}^d)$. The derivatives $\nabla_x F$ and $\nabla_u F$ exist and belong in $\mathcal{C}([0,T]; L^\infty(\mathcal{D}\times\mathbb{R}^d);\mathbb{R}^d) \cap \mathcal{C}([0,T]\times \overline{\mathcal{D}} \times \mathbb{R}^d;\mathbb{R}^d) \cap L^2(Q_T;\mathbb{R}^d)$. By continuity up to $\partial\mathcal{D}$, a trace on Σ_T exists for those functions in $L^2(\Sigma_T;\mathbb{R}^d)$.

Moreover $Hess_{x,u}(F), Hess_{u,u}(F) \in L^2(Q_T;\mathbb{R}^{2d})$.

The proof of Theorem 2.1 is divided in the three following sections:

- We prove that F has derivatives w.r.t. x and u that can be extended up to the boundary Σ_T and have finite $L^2(\Sigma_T)$ norm, we will make use of the probabilistic form of F in (1.7). In section 4, we show the regularity of the flow of the free Lagrangian process (in the sens of Bouleau Hirsch) and apply this result to prove the existence of the first order derivatives of F (see Lemma 4.6).
- In section 5, we show the L^2 regularity of the Hessians of F using a variational approach on the PDE (1.7) (see Corollary 5.5).
- In section 6, we extend some results of [Bossy and Jabir, 2015] on the semi group of the confined Langevin process with a drift.

2.2 Begining of the proof of main Theorem 1.6

Let us start with the weak error term

$$\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0})$$

for a given test function ψ.

From the definition of the function F in (1.7), we have

$$
\begin{aligned}
\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) &= \mathbb{E}F(0, X_0, U_0) - \mathbb{E}F(T, \bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(t_{i+1}, \bar{X}_{t_{i+1}}^{X_0,U_0}, \bar{U}_{t_{i+1}}^{X_0,U_0})\right) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})\right).
\end{aligned}
$$

Let us explain the last equality. The function F is continuous with respect to its three variables (t, x, u) (see Lemma 4.5). So if t_{i+1} is not a collision instant, then the scheme $(\bar{X}_t, \bar{U}_t)_{0\leq t\leq T}$ is continuous as time t_{i+1}, so the passage from the second to the third line in the previous equality is obvious. If at t_{i+1} a collision takes place, then

$$
\bar{X}_{t_{i+1}^-}^{X_0,U_0} = \bar{X}_{t_{i+1}}^{X_0,U_0} = 0
$$

and

$$
\bar{U}_{t_{i+1}}^{X_0,U_0} = -\bar{U}_{t_{i+1}^-}^{X_0,U_0},
$$

and since F satisfies the boundary specular condition, then we obtain once more the equality.

Now the collision times are introduced via the function $t \mapsto \nu(t)$ in (2.1) as follows:

$$
\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})\right).
\end{aligned}
$$

From the definition (2.1), $\nu(t_{i+1}^-) = t_i$ if there is no collision inside the period (t_i, t_{i+1}), otherwise $\nu(t_{i+1}^-) = \theta_i \neq t_i$. If no collision takes place, then by the continuity of F the first two sums of the r.h.s. are zero. If a collision does take place, then by the specular condition on F, the second term of the r.h.s. is zero. So the previous equality becomes:

$$
\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1}\left(F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0})\right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1}\left(F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})\right).
\end{aligned}
\tag{2.3}
$$

The first sum in the r.h.s can be seen as the contribution to the error of the discretized process before the jump on the time-step $[t_i, t_{i+1}]$, while the second sum is the contribution to the error of the process after the collision.

We continue the proof of the main theorem in Section 3, with the help of Theorem 2.1.

Before that, we end this section with the estimation of a bound for the L^∞ norm of the density of the confined time discretized process. In [Bossy and Jabir, 2011], it is shown that the confined Lagrangian process (1.1) admits an explicit density. Following the same arguments, we exhibit a transition density for the discretized confined Brownian primitive (i.e. $b \equiv 0$):

Lemma 2.2. *Under ($H_{\text{Weak Error}}$)-(i), the process solution to the system (2.2) with drift $b \equiv 0$ has a bounded density $p^c(t,\zeta,\eta)$, bounded by $2\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}$.*

Proof. Starting from a given (x,u) in $\mathcal{D}\times\mathbb{R}^d$, the process (2.2) without drift can be written as:

$$\begin{cases} \bar{x}_t = x + \displaystyle\int_0^t \bar{u}_{\eta(s)} \operatorname{sign}\left(\bar{x}_{\eta(s)} + (s-\eta(s))\bar{u}_{\eta(s)}\right) ds \\ \bar{u}_t = u + \sigma W_t^0 - 2 \displaystyle\sum_{0<s\leq t} \bar{u}_{s^-} \mathbb{1}_{\bar{x}_s=0} \end{cases} \tag{2.4}$$

where $(W_t^0)_{t\geq 0}$ is a standard Brownian motion. Following the arguments in [Bossy and Jabir, 2011], we introduce the continuous time-discretized free Langevin process with $b=0$:

$$\begin{cases} \bar{Z}_t = x + \displaystyle\int_0^t \bar{V}_{\eta(s)}\, ds \\ \bar{V}_t = u + \sigma W_t. \end{cases} \tag{2.5}$$

The position process $\bar{Z}_t$ can be rewritten as

$$\bar{Z}_t = x + ut + \sigma \sum_{i\geq 0} W_{t_i\wedge t}(t_{i+1}\wedge t - t_i \wedge t).$$

Since $(W_t)_{t\geq 0}$ is a Gaussian process, then $(\bar{Z}_t,\bar{V}_t)_{t\geq 0}$ is also a Gaussian process due to the fact that it can be written as a linear combination of random variables sampled from a Gaussian process at different instants. In particular, there is a Gaussian transition density for the time-discretized Langevin process with no drift, denoted as $\bar{p}^L$ (see Section 2 for the explicit expression for $\bar{p}^L$.)

Define $S_t = \operatorname{sign}(Z_t)_+$ to be the càdlàg modification of the process $(\operatorname{sign}(Z_t))_{0\leq t\leq T}$ and set

$$(\bar{X}_t^c, \bar{U}_t^c) = (|\bar{Z}_t|, S_t\bar{V}_t). \tag{2.6}$$

Then, by the Itô's formula, we get

$$\bar{U}_t^c = u + \int_0^t S_{s^-}\, d\bar{V}_s + \sum_{0<s\leq t} \bar{V}_s \Delta S_s = u + \sigma \int_0^t S_{s^-}\, dW_s + \sum_{0<s\leq t} \bar{V}_s \Delta S_s.$$

Since $\langle \int_0^\cdot S_{s^-}\, dW_s\rangle_t = t$, by Lévy's representation theorem, the process $(W_t^c = \int_0^t S_{s^-}\, dW_s, t\geq 0)$ is a Brownian motion. Also, by continuity of the process $(\bar{V}_t)_{0\leq t\leq T}$, for any $t\in[0,T]$, $\bar{U}_{t-}^c = \bar{V}_{t-}S_{t-} = \bar{V}_t S_{t-}$. Consider a time interval $[t_i,t_{i+1}]$ such that $t_i < \theta_i < t_{i+1}$, then if $S_{\eta(t)} > 0$, then $S_{\theta_i^-} > 0$ implying that $\Delta S_{\theta_i} = -2 = -2S_{\theta_i^-}$ and if $S_{\eta(t)} < 0$, then $S_{\theta_i^-} < 0$ resulting in $\Delta S_{\theta_i} = 2 = -2S_{\theta_i^-}$. These considerations give that $\bar{V}_{\theta_i}\Delta S_{\theta_i} = -2U^c_{\theta_i^-}$, and finally, we have that

$$\bar{U}_t^c = u + \sigma W_t^c - 2\sum_{0<s\leq t} \bar{U}^c_{s^-} \mathbb{1}_{\{\bar{X}_s^c = 0\}}.$$

Considering that $(Z_t^c)_{0\leq t\leq T}$ change it sign a finite number of time, it admits a regularity C^1 by parts. We obtain that

$$\bar{X}_t^c = |\bar{Z}_t| = x + \int_0^t \operatorname{sign}(\bar{Z}_s)\bar{V}_{\eta(s)}\, ds.$$

From (2.5), we notice that

$$\bar{Z}_t = \bar{Z}_{\eta(t)} + (t-\eta(t))\bar{V}_{\eta(t)} = \operatorname{sign}(\bar{Z}_{\eta(t)})\left(|\bar{Z}_{\eta(t)}| + (t-\eta(t))\operatorname{sign}(\bar{Z}_{\eta(t)})\bar{V}_{\eta(t)}\right),$$

since $\operatorname{sign}(ab) = \operatorname{sign}(a)\operatorname{sign}(b)$. So,

$$\bar{Z}_t = S_{\eta(t)} \left(\bar{X}^c_{\eta(t)} + (t-\eta(t))\bar{U}^c_{\eta(t)} \right)$$

and

$$\begin{aligned}
\bar{X}^c_t &= x + \int_0^t \operatorname{sign}\left(S_{\eta(s)} \left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)} \right)\right) \bar{V}_{\eta(s)}\, ds \\
&= x + \int_0^t \operatorname{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)} \right) S_{\eta(s)}\bar{V}_{\eta(s)}\, ds \\
&= x + \int_0^t \operatorname{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)} \right) \bar{U}^c_{\eta(s)}\, ds
\end{aligned}$$

obtaining finally:

$$\begin{cases}
\bar{X}^c_t = x + \displaystyle\int_0^t \operatorname{sign}\left(\bar{X}^c_{\eta(s)} + (s-\eta(s))\bar{U}^c_{\eta(s)} \right) \bar{U}^c_{\eta(s)}\, ds \\
\bar{U}^c_t = u + \sigma W^c_t - 2 \displaystyle\sum_{0<s\le t} \bar{U}^c_{s-} \mathbb{1}_{\{\bar{X}^c_s = 0\}}.
\end{cases}$$

This shows that $(\bar{X}^c_t, \bar{U}^c_t)_{0\le t\le T}$ defined as (2.6) is equal in law to the solution of (2.4) $(\bar{x}_t, \bar{u}_t)_{0\le t\le T}$. This also implies that $(|\bar{u}_t|)_{t\ge 0}$ is equal in distribution to $(|u+W_t|)_{t\ge 0}$. Furthermore, for any measurable and bounded function $h\colon \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$:

$$\mathbb{E}h(\bar{x}_t, \bar{u}_t) = \mathbb{E}\left(h(\bar{Z}_t, \bar{V}_t)\mathbb{1}_{\{\bar{Z}_t>0\}}\right) + \mathbb{E}\left(h(-\bar{Z}_t, -\bar{V}_t)\mathbb{1}_{\{\bar{Z}_t<0\}}\right).$$

as $\{\bar{Z}_t = 0\}$ is negligible. The transition density of the discretized reflected process $\bar{p}^c\colon (\mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{R})) \times (\mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{R}) \to \mathbb{R}$ is then equal to

$$\bar{p}^c(0,x,u;t;\xi,\zeta) = \bar{p}^L(0,x,u;t;\xi,\zeta) + \bar{p}^L(0,x,u;t;-\xi,-\zeta)$$

where $\bar{p}^L$ is the transition density of the time-discretized free process (2.5) computed in Lemma 2.1 of the appendix section 2.

Now we consider the hypothesis $(H_{\textit{Weak Error}})$-$(i)$, and μ_0 the density of the initial random variable (X_0, U_0). The density of $(\bar{x}^{X_0,U_0}_t, \bar{u}^{X_0,U_0}_t)_{0\le t\le T}$ writes

$$\begin{aligned}
p^c(t;\xi,\zeta) &= \int_{\mathbb{R}\times\mathbb{R}^+} \bar{p}^c(0;x,u;t;\xi,\zeta)\mu_0(x,u)\, dxdu \\
&= \int_{\mathbb{R}\times\mathbb{R}^+} \left(\bar{p}^L(0;x,u;t;\xi,\zeta) + \bar{p}^L(0;x,u;t;-\xi,-\zeta)\right) \mu_0(x,u)\, dxdu \\
&= \int_{\mathbb{R}\times\mathbb{R}^+} \left(p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(\xi-(x+tu), \zeta-u) + p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(-\xi-(x+tu), -\zeta-u)\right) \mu_0(x,u)\, dxdu,
\end{aligned}$$

where $p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}$ denotes the centered Gaussian density with covariance $\Sigma_{t,\Delta t,\eta(t)}$ computed in Lemma 2.1. Then

$$\begin{aligned}
&p^c(t;\xi,\zeta) \\
&\le \|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})} \int_{\mathbb{R}\times\mathbb{R}} \left(p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(\xi-(x+tu), \zeta-u) + p_{\mathcal{N}(0,\Sigma_{t,\Delta t,\eta(t)})}(-\xi-(x+tu), -\zeta-u)\right) dxdu \\
&\le 2\,\|\mu_0\|_{L^\infty(\mathcal{D}\times\mathbb{R})}.
\end{aligned}$$

■

3 Weak error estimation

In this section we prove the main theorem 1.6. In order to simplify the presentation, we give the proof for the dimension $d = 1$ and in order to better understand the various definitions for the errors that have been introduced we refer to the diagram 2 in the Appendix section 1.

The contributions to the error (1.8) mainly come from the discretisation of the drift of the position process and of the drift of the velocity process. Each of these components will be separated in the terms before the collision with the reflecting boundary and after the collision. As seen in the sketched proof in Section 3.1, the Itô's formula is applied two times in the terms of the decomposition of the error (2.3). Those terms involve the function F in (1.7) which does not have apriori a sufficient regularity. To overcome this difficulty, we first smooth the function F for each variables (t, x, u), with the mollifying sequences $(\beta_k, \rho_l, g_m)_{k,l,m\geq 1}$.

Smooth approximation of F. We construct $(\beta_k)_{k\geq 1}$, $(\rho_l)_{l\geq 1}$ and $(g_m)_{m\geq 1}$, some positive approximations to the identity such that:

$$\mathrm{Supp}(\beta_k) \subset \left(0, \frac{T}{k}\right), \qquad \mathrm{Supp}(\rho_l) \subset \left(-\frac{1}{l}, 0\right) \qquad \text{and} \qquad \mathrm{Supp}(g_m) = \mathbb{R}. \tag{3.1}$$

For $(\beta_k)_{k\geq 1}$, we consider the function $t \mapsto \beta(t)$ defined on $\mathbb{R}$ by

$$\beta(t) = \begin{cases} \exp\left(-\dfrac{1}{t(T-t)}\right) & \text{for } t \in (0,T)\ , \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.2}$$

Then for $k \geq 1$, we set $\beta_k(t) = C'k\beta(kt)$ where C' is such that $\displaystyle\int_{[0,T]} \beta(t)\, dt = \frac{1}{C'}$. With the choice for the support of β to be included in $(0, T)$, we have that any convolution on $[0, T]$ is zero at $t = 0$. For example consider the function $h\colon [0, T] \mapsto \mathbb{R}$, then the function $\hat{h}\colon s \mapsto \displaystyle\int_{[0,T]} \beta_k(s-\tau)h(\tau)\, d\tau$ is such that for any $k \geq 1$, $\hat{h}(0) = 0$. We can easily see this in the following graph where we consider $T = 1$, $k = 10$ and $h\colon s \mapsto \mathbb{1}_{[0,1]}(s)(2-s)$.

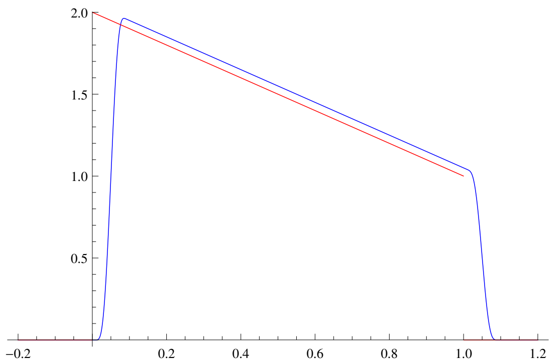


Figure 1.1: Convolution (in blue) on $[0, T]$ between $s \mapsto h(s) = \mathbb{1}_{[0,1]}(s)(2-s)$ (in red) and mollifier β_k

For $(\rho_l)_{l\geq 1}$, we consider the generating function $x \mapsto \rho(x)$, defined on $\mathbb{R}$ by

$$\rho(x) = \begin{cases} \exp\left(-\dfrac{1}{x(-1-x))}\right) & \text{for } x \in (-1,0)\,, \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.3}$$

Then for any $l \geq 1$, $\rho_l(x) = Cl\rho(lx)$ where C is such that $\displaystyle\int_{\mathbb{R}} \rho(x)\,dx = \frac{1}{C}$.

For the sequence $(g_m)_{m\geq 1}$, we choose to use the Gaussian kernel $u \mapsto g(u)$ on $\mathbb{R}$ with

$$g(u) = \frac{1}{\sqrt{(2\pi)}} \exp\left(-\frac{u^2}{2}\right) \tag{3.4}$$

by taking $g_m(u) = mg(mu)$. We obtain the smooth function: $\forall (t,x,u) \in \overline{Q_T}$,

$$F_{k,l,m}(t,x,u) = \int_{Q_T} F(\tau,y,v)\beta_k(t-\tau)\rho_l(x-y)g_m(u-v)\,d\tau dy dv. \tag{3.5}$$

We mention again that for the choice of the mollifying sequence $(\beta_k)_{k\geq 1}$ to have support on $\left(0, \frac{T}{k}\right)$, we obtain that

$$\forall (x,u) \in \mathcal{D} \times \mathbb{R}, \quad F_{k,l,m}(0,x,u) = 0\,.$$

We denote by L the infinitesimal generator for the process $(X_t, U_t)_{0\leq t\leq T}$:

$$L = u\partial_x + b(x,u)\partial_u + \frac{\sigma^2}{2}\partial^2_{uu}.$$

As a corollary of Lemma 5.2 in Section 5, we have

Corollary 3.1. *The smooth function $F_{k,l,m}$ defined on $\overline{Q_T}$ satisfies the following equality for any (t,x,u) in the interior of Q_T:*

$$\left(\frac{\partial}{\partial t} + L\right) F_{k,l,m}(t,x,u) = R_{k,l,m}[F](t,x,u). \tag{3.6}$$

with

$$R_{k,l,m}[F](t,x,u) = R^{\mathrm{Sp}}_{k,l,m}[F](t,x,u) + R^{\mathrm{Tm}}_{k,l,m}[F](t,x,u)$$

where

$$\begin{aligned} R^{\mathrm{Sp}}_{k,l,m}[F](t,x,u) &:= (\partial_x F * (u g_m \rho_l \beta_k))(t,x,u) + b(x,u)\cdot((\partial_u F * (g_m\rho_l\beta_k))(t,x,u)) \\ &\quad - ((b\cdot\partial_u F) * (g_m\rho_l\beta_k))(t,x,u) \end{aligned}$$

$$R^{\mathrm{Tm}}_{k,l,m}[F](t,x,u) := \beta_k(t)F(0,\cdot,\cdot) * (g_m\rho_l)(x,u).$$

Proof. We apply Lemma 5.2 by noticing that for any $(t,x,u) \in Q_T$, $f(t,x,u) = F(T-t,x,u)$. We have by the definition of $f_{k,l,m}$ in (5.4) for any $(\tau,y,v) \in Q_T$ and since $\widetilde{\beta}_k(t) = \beta_k(-t)$

$$\begin{aligned} f_{k,l,m}(T-\tau,y,v) &= \int_{Q_T} f(s,x,u)\widetilde{\beta}_k(T-\tau-s)\rho_l(y-x)g_m(v-u)\,ds dx du \\ &= \int_{Q_T} f(T-t,x,u)\widetilde{\beta}_k(t-\tau)\rho_l(y-x)g_m(v-u)\,dt dx du \\ &= \int_{Q_T} F(t,x,u)\beta_k(\tau-t)\rho_l(y-x)g_m(v-u)\,dt dx du = F_{k,l,m}(\tau,y,v)\,, \end{aligned} \tag{3.7}$$

where the change of variable $s \to T-t$ was performed and we obtain that $\partial_t f_{k,l,m}(T-t,x,y) = -\partial_t F_{k,l,m}(t,x,y)$. Now we consider the rest term $R^{\mathrm{Tm}}_{k,l,m}[f]$ of Lemma 5.2 and have

$$\begin{aligned} R^{\mathrm{Tm}}_{k,l,m}[f](T-\tau,y,v) &= \widetilde{\beta}_k((T-\tau)-T)f_{l,m}(T,y,v) \\ &= \widetilde{\beta}_k(-\tau)F_{l,m}(0,y,v) = \beta_k(\tau)F_{l,m}(0,y,v) \\ &= R^{\mathrm{Tm}}_{k,l,m}[F](\tau,y,v) \end{aligned}$$

with $F_{l,m}(0,\cdot,\cdot) = F(0,\cdot,\cdot) * (g_m\rho_l)(\cdot,\cdot)$.

From these equalities it is straightforward to conclude the result of the lemma. ∎

$R^{\mathrm{Sp}}_{k,l,m}$ denotes mainly the spatial contribution to the regularization error. Since we choose ψ in $\mathcal{C}^{1,1}_c(\mathcal{D}\times\mathbb{R};\mathbb{R})$, applying Theorem 2.1, we obtain that $\partial_x F$ and $\partial_u F$ are well defined and belong in $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R});\mathbb{R})\cap\mathcal{C}([0,T]\times\overline{D}\times\mathbb{R};\mathbb{R})$. Later in Lemma 5.3 we prove that $R^{\mathrm{Sp}}_{k,l,m}$ converges uniformly to 0 as k, l and m go to infinity.

$R^{\mathrm{Tm}}_{k,l,m}$ is mostly a temporal contribution to the regularization error. We prove that $\int_0^T R^{\mathrm{Tm}}_{k,l,m}[F]$ converges uniformly toward $F(0,\cdot,\cdot)$ as k, l and m go to infinity.

Now we can go back to the error decomposition made in (2.3)

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0})-\mathbb{E}\psi(\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0})\\
&=\mathbb{E}\sum_{i=0}^{n-1}\Big(F(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})-F(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\
&\quad+\mathbb{E}\sum_{i=0}^{n-1}\Big(F(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})-F(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big),
\end{aligned}$$

and introduce the smooth solution and pick k such that $\mathrm{Supp}(\beta_k)\subset(0,\Delta t\wedge\varepsilon_0)$ with ε_0 defined in $(H_{\textit{Weak Error}})$-$(i)$:

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0})-\mathbb{E}\psi(\bar{X}_T^{X_0,U_0},\bar{U}_T^{X_0,U_0})\\
&=\mathbb{E}\sum_{i=0}^{n-1}\Big(F_{k,l,m}(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})-F_{k,l,m}(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\
&\quad+\mathbb{E}\sum_{i=0}^{n-1}\Big(F_{k,l,m}(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})-F_{k,l,m}(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big)\\
&\quad+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\
&\quad+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big).
\end{aligned}\tag{3.8}$$

3.1 On the error terms introduced by regularizing the solution

The regularisation in time and space components introduce some errors that we analyse. Special care is taken for the time regularisation since it introduced a term $R^{\mathrm{Tm}}_{k,l,m}$ that cannot be bounded uniformly in k. We denote by $\mathrm{Reg}_{k,l,m}$ the term:

$$\begin{aligned}
\mathrm{Reg}_{k,l,m}:=&\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t_i,\bar{X}^{X_0,U_0}_{t_i},\bar{U}^{X_0,U_0}_{t_i})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\Big)\\
&+\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(\nu(t^-_{i+1}),\bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})},\bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\Big)-\mathbb{E}\sum_{i=0}^{n-1}\Big((F-F_{k,l,m})\,(t^-_{i+1},\bar{X}^{X_0,U_0}_{t^-_{i+1}},\bar{U}^{X_0,U_0}_{t^-_{i+1}})\Big).
\end{aligned}\tag{3.9}$$

Assuming no collision takes place on the first discretisation interval

If no collision takes place on (t_0, t_1), we have that

$$\begin{aligned}\mathrm{Reg}_{k,l,m} &= \mathbb{E}\sum_{i=1}^{n-1}\Big(\left(F-F_{k,l,m}\right)(t_i, \bar{X}^{X_0,U_0}_{t_i}, \bar{U}^{X_0,U_0}_{t_i})\Big) - \mathbb{E}\sum_{i=1}^{n-1}\left(\left(F-F_{k,l,m}\right)(\nu(t^-_{i+1}), \bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})}, \bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\right)\\ &+ \mathbb{E}\sum_{i=1}^{n-1}\left(\left(F-F_{k,l,m}\right)(\nu(t^-_{i+1}), \bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})}, \bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\right) - \mathbb{E}\sum_{i=0}^{n-1}\left(\left(F-F_{k,l,m}\right)(t^-_{i+1}, \bar{X}^{X_0,U_0}_{t^-_{i+1}}, \bar{U}^{X_0,U_0}_{t^-_{i+1}})\right)\\ &+ \mathbb{E}F(0, X_0, U_0)\end{aligned} \tag{3.10}$$

Assuming a collision takes place on the first discretisation interval

If a collision takes place on (t_0, t_1), we have that

$$\begin{aligned}\mathrm{Reg}_{k,l,m} &= \mathbb{E}\sum_{i=1}^{n-1}\Big(\left(F-F_{k,l,m}\right)(t_i, \bar{X}^{X_0,U_0}_{t_i}, \bar{U}^{X_0,U_0}_{t_i})\Big) - \mathbb{E}\sum_{i=0}^{n-1}\left(\left(F-F_{k,l,m}\right)(\nu(t^-_{i+1}), \bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})}, \bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})^-})\right)\\ &+ \mathbb{E}\sum_{i=0}^{n-1}\left(\left(F-F_{k,l,m}\right)(\nu(t^-_{i+1}), \bar{X}^{X_0,U_0}_{\nu(t^-_{i+1})}, \bar{U}^{X_0,U_0}_{\nu(t^-_{i+1})})\right) - \mathbb{E}\sum_{i=0}^{n-1}\left(\left(F-F_{k,l,m}\right)(t^-_{i+1}, \bar{X}^{X_0,U_0}_{t^-_{i+1}}, \bar{U}^{X_0,U_0}_{t^-_{i+1}})\right)\\ &+ \mathbb{E}F(0, X_0, U_0)\end{aligned} \tag{3.11}$$

In both cases we denote

$$\mathrm{Reg}_{k,l,m} = \epsilon^{\mathrm{Reg}}_{k,l,m} + \mathbb{E}F(0, X_0, U_0)\,. \tag{3.12}$$

For any $i \in \{0, \cdots, n-1\}$, we denote the error obtained before a collision as:

$$\epsilon_{\mathrm{BR}}(i) := \mathbb{E}\left[\left(F_{k,l,m}(t_i, \bar{X}^{X_0,U_0}_{t_i}, \bar{U}^{X_0,U_0}_{t_i}) - F_{k,l,m}(\theta_i, \bar{X}^{X_0,U_0}_{\theta_i}, \bar{U}^{X_0,U_0}_{\theta_i^-})\right)\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}}\right] \tag{3.13}$$

and after the collision as

$$\epsilon_{\mathrm{AR}}(i) := \mathbb{E}\left[\left(F_{k,l,m}(\theta_i, \bar{X}^{X_0,U_0}_{\theta_i}, \bar{U}^{X_0,U_0}_{\theta_i}) - F_{k,l,m}(t^-_{i+1}, \bar{X}^{X_0,U_0}_{t^-_{i+1}}, \bar{U}^{X_0,U_0}_{t^-_{i+1}})\right)\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}}\right]. \tag{3.14}$$

If no collision occurs on (t_i, t_{i+1}) the error is denoted as

$$\epsilon_{\mathrm{NoR}}(i) := \mathbb{E}\left[\left(F_{k,l,m}(t_i, \bar{X}^{X_0,U_0}_{t_i}, \bar{U}^{X_0,U_0}_{t_i}) - F_{k,l,m}(t^-_{i+1}, \bar{X}^{X_0,U_0}_{t^-_{i+1}}, \bar{U}^{X_0,U_0}_{t^-_{i+1}})\right)\mathbb{1}_{\{\theta_i = t_i\}}\right]. \tag{3.15}$$

The ϵ_{BR}, ϵ_{AR}, ϵ_{NoR} are the terms that we develop through an application of Itô's formula. On each sub-intervals $[t_i, \theta_i))$, we introduce the partial differential operator

$$\mathcal{L}_{\mathrm{BR}}h(t,x,u) := \left(\bar{U}_{t_i}\partial_x h + b(\bar{X}_{t_i}, \bar{U}_{t_i})\partial_u h + \frac{\sigma^2}{2}\partial^2_{uu}h\right)(t,x,u),$$

and on the interval $[\theta_i, t_{i+1})$ we define:

$$\mathcal{L}_{\mathrm{AR}}h(t,x,u) := \left(-\bar{U}_{t_i}\partial_x h + b(\bar{X}_{\nu(t)}, \bar{U}_{\nu(t)})\partial_u h + \frac{\sigma^2}{2}\partial^2_{uu}h\right)(t,x,u),$$

if no collision occurs on (t_i, t_{i+1}), for any $h \in C^{1,1,2}(Q_t)$, we have the operator

$$\mathcal{L}_{\text{NoR}} h(t,x,u) := \left(\bar{U}_{t_i} \partial_x h + b(\bar{X}_{t_i}, \bar{U}_{t_i}) \partial_u h + \frac{\sigma^2}{2} \partial^2_{uu} h \right)(t,x,u),$$

where $h \in C^{1,1,2}(Q_t)$. The subscript BR signifies "before reflection", AR signifies "after reflection" and NoR signifies "no reflection". The $\text{sign}(\bar{Y}_t)$ dependency is in fact a constant term such that

$$\text{sign}(\bar{Y}_t) = \begin{cases} 1, & \forall t \in [t_i, \theta_i),\ \theta_i \neq t_i, & \text{BR} \\ -1, & \forall t \in [\theta_i, t_{i+1}),\ \theta_i \neq t_i, & \text{AR} \\ 1, & \forall t \in [t_i, t_{i+1}),\ \theta_i = t_i. & \text{NoR} \end{cases}$$

or to be more explicit, $\text{sign}(\bar{Y}_t)$ equals 1 in $\mathcal{L}_{\text{BR}}$ and $\mathcal{L}_{\text{NoR}}$ and -1 for $\mathcal{L}_{\text{AR}}$. It can be seen that the differential operator $\mathcal{L}_{\text{BR}}$ and $\mathcal{L}_{\text{NoR}}$ (before a collision or if no collision occurs) are similar, so the results from one apply to the other if the time interval of application is adjusted accordingly.

By applying the Itô formula to the first two terms of (3.8), we obtain that:

$$\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \mathbb{E}\sum_{i=0}^{n-1} \left(F_{k,l,m}(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) - F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0}) \right) \\
&\quad + \mathbb{E}\sum_{i=0}^{n-1} \left(F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) - F_{k,l,m}(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0}) \right) + \text{Reg}_{k,l,m} \\
&= \sum_{i=0}^{n-1} (\epsilon_{\text{BR}}(i) + \epsilon_{\text{AR}}(i) + \epsilon_{\text{NoR}}(i)) + \epsilon_{k,l,m}^{\text{Reg}} + \mathbb{E}F(0, X_0, U_0) \\
&= -\sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\partial_t + \mathcal{L}_{\text{BR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0}) ds \right] \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\partial_t + \mathcal{L}_{\text{AR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \right] \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{\theta_i = t_i\}} \int_{t_i}^{t_{i+1}} (\partial_t + \mathcal{L}_{\text{NoR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \right] + \epsilon_{k,l,m}^{\text{Reg}} + \mathbb{E}F(0, X_0, U_0).
\end{aligned} \tag{3.16}$$

The stochastic integrals terms are actually martingales since by Theorem 2.1, $\partial_u F \in L^\infty(Q_T)$. Since $F_{k,l,m}$ is a solution to the equation (3.6):

$$
\begin{aligned}
&\mathbb{E}\psi(X_T^{X_0,U_0}, U_T^{X_0,U_0}) - \mathbb{E}\psi(\bar{X}_T^{X_0,U_0}, \bar{U}_T^{X_0,U_0}) \\
&= \sum_{i=0}^{n-1} (\epsilon_{\mathrm{BR}}(i) + \epsilon_{\mathrm{AR}}(i) + \epsilon_{\mathrm{NoR}}(i)) + \epsilon_{k,l,m}^{\mathrm{Reg}} + \mathbb{E}F(0, X_0, U_0) \\
&= \sum_{i=0}^{n-1} \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (L - \mathcal{L}_{\mathrm{BR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&\quad + \sum_{i=0}^{n-1} \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (L - \mathcal{L}_{\mathrm{AR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&\quad + \sum_{i=0}^{n-1} \mathbb{E}\mathbb{1}_{\{\theta_i = t_i\}} \int_{t_i}^{t_{i+1}} (L - \mathcal{L}_{\mathrm{NoR}}) F_{k,l,m}(s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} R_{k,l,m}^{\mathrm{Sp}}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&\quad - \sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} R_{k,l,m}^{\mathrm{Tm}}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds + \mathbb{E}F(0, X_0, U_0) + \epsilon_{k,l,m}^{\mathrm{Reg}}\,.
\end{aligned}
\tag{3.17}
$$

Remark 3.2. *By Theorem 2.1, F is in $W^{(1,1),2}(Q_T)$. A generalized Ito's Lemma (see e.g. Theorem 1, page 122 of [Krylov, 1980]) with the extension for unbounded domains and hypo-elliptic diffusions, should have been applied in this part of the proof, instead of regularising F.*

We now present a lemma that gives the convergence of the various terms that compose the error obtained by regularization.

Lemma 3.3. *We have that*

(i) $\left|\epsilon_{k,l,m}^{Reg}\right| \xrightarrow{k,l,m\to\infty} 0$

(ii) $\left|\sum_{i=0}^{n-1} \mathbb{E} \int_{t_i}^{t_{i+1}} R_{k,l,m}^{Tm}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds - EF(0, X_0, U_0)\right| \xrightarrow{k,l,m\to\infty} 0$

Proof. **Convergence *(i)*.**

According to the Lemma 4.5, F is continuous and bounded on $\overline{Q_T}$ and in fact we can extend naturally F as a continuous, bounded function on $[0, T] \times \mathbb{R} \times \mathbb{R}$ (for an example of such an extension on the whole domain see the calculations (4.25) in Section 4 and take $F(t, x, u) = f(T - t, x, u)$).

We recall that if a collision occurs on the first interval (t_0, t_1), that we have that

$$
\begin{aligned}
\epsilon_{k,l,m}^{\mathrm{Reg}} &= \mathbb{E} \sum_{i=1}^{n-1} \Big((F - F_{k,l,m})\,(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0}) \Big) - \mathbb{E} \sum_{i=0}^{n-1} \Big((F - F_{k,l,m})\,(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)^-}^{X_0,U_0}) \Big) \\
&+ \mathbb{E} \sum_{i=0}^{n-1} \Big((F - F_{k,l,m})\,(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0}) \Big) - \mathbb{E} \sum_{i=0}^{n-1} \Big((F - F_{k,l,m})\,(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0}) \Big)
\end{aligned}
\tag{3.18}
$$

and we apply Lemma 1.6, in the Appendix section 1, which states we have that $F_{k,l,m}$ converges uniformly on any compact of $(0, T] \times \mathbb{R} \times \mathbb{R}$. In our case, we consider the compact $[\varepsilon_0 \wedge t_1, T] \times \mathbb{R} \times \mathbb{R}$.

By condition $(H_{\textit{Weak Error}})$-$(i)$ (see Remark 1.5), the first collision time $\nu(t_1^-)$ is such that $\nu(t_1^-) \geq \varepsilon_0$, therefore the random variables $F_{k,l,m}(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0})$ (this term considered only for $i \geq 1$), $F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)-}^{X_0,U_0})$, $F_{k,l,m}(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})$ and $F_{k,l,m}(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})$ converge almost surely to $F(t_i, \bar{X}_{t_i}^{X_0,U_0}, \bar{U}_{t_i}^{X_0,U_0})$, $F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)-}^{X_0,U_0})$, $F(\nu(t_{i+1}^-), \bar{X}_{\nu(t_{i+1}^-)}^{X_0,U_0}, \bar{U}_{\nu(t_{i+1}^-)}^{X_0,U_0})$ and, respectively, $F(t_{i+1}^-, \bar{X}_{t_{i+1}^-}^{X_0,U_0}, \bar{U}_{t_{i+1}^-}^{X_0,U_0})$. Since F is a bounded function, then $\epsilon_{k,l,m}^{\text{Reg}}$ goes to zero as k, l, m go to infinity by the Dominated Convergence Theorem.

Similar arguments apply if there is no collision on the first interval (t_0, t_1).

Convergence *(ii)*.

Since k has been chosen such that $\text{Supp}(\beta_k) \subset (0, \Delta t \wedge \varepsilon_0)$, and since no collision occurs on $(0, \Delta t \wedge \varepsilon_0) \equiv (t_0, t_1 \wedge \varepsilon_0)$ (see Remark 1.5) then we have that

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} R_{k,l,m}^{\text{Tm}}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds = \int_0^T R_{k,l,m}^{\text{Tm}}[F](s, \bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&= \int_0^T \beta_k(s) F(0,\cdot,\cdot) * (\rho_l g_m)(\bar{X}_s^{X_0,U_0}, \bar{U}_s^{X_0,U_0})\, ds \\
&= \int_0^{\Delta t \wedge \varepsilon_0} \beta_k(s) F(0,\cdot,\cdot) * (\rho_l g_m)(X_0 + sU_0, U_0)\, ds\,.
\end{aligned}
$$

By uniform convergence arguments of convolutions used in the previous section we have that $F(0,\cdot,\cdot) * (\rho_l g_m)(X_0 + sU_0, U_0)$ converges a.s. to $F(0, X_0 + sU_0, U_0)$. We introduce the function $g\colon [0,T] \mapsto \mathbb{R}$, such that for any $s \in [0,T]$ $g(s) = F(0, X_0 + sU_0, U_0)$. By Lemma 4.5, we have that F is continuous on Q_T, therefore g is a continuous function on $[0,T]$.

For any $\epsilon > 0$, there exists $\delta > 0$ such that $|g(0) - g(s)| < \varepsilon$, for any $s \in (0, \delta)$.

We recall that $\text{Supp}(\beta_k) \in (0, \frac{T}{k})$ so the previous equality becomes

$$
\begin{aligned}
&\int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds - F(0, X_0, U_0) = \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) g(s)\, ds - g(0) \\
&= \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)(g(s) - g(0))\, ds
\end{aligned}
$$

so for every k such that $\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0 \leq \delta$, we obtain that

$$
\left| \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)(g(s) - g(0))\, ds \right| \leq \varepsilon \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)\, ds = \varepsilon\,.
$$

Thus, $\int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds$ converges almost surely towards $F(0, X_0, U_0)$. As F is a bounded function therefore

$$
\left| \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s) F(0, X_0 + sU_0, U_0)\, ds \right| \leq \|F\|_{L^\infty(Q_T)} \int_0^{\frac{T}{k} \wedge \Delta t \wedge \varepsilon_0} \beta_k(s)\, ds = \|F\|_{L^\infty(Q_T)}
$$

then by the dominated convergence theorem, we obtain the desired result. ■

In order to simplify the writing, we remove the references to the initial conditions and write simply $(\bar{X}_t, \bar{U}_t)$ as $(\bar{X}_t^{X_0,U_0}, \bar{U}_t^{X_0,U_0})$.

For all $i \in \{0, \ldots, n-1\}$, according to the definition of $\mathcal{L}_{\text{BR}}$, the term under the first summation in the r.h.s of equality (3.17) is rewritten as:

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (L - \mathcal{L}_{\text{BR}}) F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\bar{U}_s - \bar{U}_{\eta(s)}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&\quad + \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} \left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\right) \partial_u F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&=: \epsilon_{\text{BR}}^{\bar{X}}(i) + \epsilon_{\text{BR}}^{\bar{U}}(i).
\end{aligned} \tag{3.19}$$

The third sum in the r.h.s. of equality (3.17) corresponds to the case without reflection, and it can be developed similarly to

The term under the second summation in the r.h.s. of equality (3.17) is:

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (L - \mathcal{L}_{\text{AR}}) F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\bar{U}_s - \bar{U}_{\eta(s)} \operatorname{sign}(\bar{Y}_s)) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds \\
&\quad + \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\right) \partial_u F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\bar{U}_s + \bar{U}_{\eta(s)}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&\quad + \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\right) \partial_u F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&=: \epsilon_{\text{AR}}^{\bar{X}}(i) + \epsilon_{\text{AR}}^{\bar{U}}(i).
\end{aligned} \tag{3.20}$$

We recall that for $s \in [\theta_i, t_{i+1})$, $\bar{U}_s$ is the velocity after specular reflection, so there is a change of sign at θ_i.

The error is then further decomposed with contribution from the discretization of the drift of the position process $(\bar{X}_t)_{0 \le t \le T}$ and a contribution from the drift of the velocity process $(\bar{U}_t)_{0 \le t \le T}$. We denote these errors before the reflection as $\epsilon_{\text{BR}}^{\bar{X}}(i)$, $\epsilon_{\text{BR}}^{\bar{U}}(i)$ respectively, after the reflection $\epsilon_{\text{AR}}^{\bar{X}}(i)$ and $\epsilon_{\text{AR}}^{\bar{U}}(i)$. We finally denote $\epsilon_{\text{NoR}}^{\bar{X}}(i)$ and $\epsilon_{\text{NoR}}^{\bar{U}}(i)$ the error obtained when no reflection occurs on the interval. The superscript $^{\bar{X}}$ denotes the error related to the approximation of the position of the particle while the superscript $^{\bar{U}}$ denotes the error due to the approximation of the velocity of the particle.

3.2 Contribution to the error $\epsilon^{\bar{X}}$ of the discretized drift on the position process

Contribution to the error before the reflection

We begin by developing the error produced by the discretization of position process, before reflection:

$$\begin{aligned}
\epsilon_{\text{BR}}^{\bar{X}}(i) &= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\bar{U}_s - \bar{U}_{\eta(s)})) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (s - t_i) b(\bar{X}_{t_i}, \bar{U}_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&\quad + \sigma \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (W_s - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds.
\end{aligned} \tag{3.21}$$

We consider the inner integral

$$
\begin{aligned}
&\int_{t_i}^{\theta_i} (\bar{U}_s - \bar{U}_{\eta(s)}))\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \int_{t_i}^{\theta_i} (s - t_i) b(\bar{X}_{t_i}, \bar{U}_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds + \int_{t_i}^{\theta_i} (W_s - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds.
\end{aligned}
$$

The second term of this equality is treated separately by conditioning w.r.t $\mathcal{F}_{t_i}$. For any $s \geq t_i$, the increment $W_s - W_{t_i}$ is independent to the $\sigma-$algebra $\mathcal{F}_{t_i}$, so by introducing the probability density function of the standard Gaussian random variable denoted $p_{\mathcal{N}(0,1)}$, we have:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_i}^{\theta_i} (W_s - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \mathbb{E}\left[\int_{t_i}^{\theta_i} (W_s - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma(W_s - W_{t_i}))\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \int_{t_i}^{\theta_i} \sqrt{s-t_i}\, ds \int_{\mathbb{R}} w\partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma\sqrt{s-t_i}w) p_{\mathcal{N}(0,1)}(w) dw.
\end{aligned}
$$

The integral can be transformed to obtain a derivative of the Gaussian density:

$$
\begin{aligned}
&\int_{\mathbb{R}} w\partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s - t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s - t_i) + \sigma\sqrt{s - t_i}w) p_{\mathcal{N}(0,1}(w)\, dw \\
&= -\int_{\mathbb{R}} \partial_x F_{k,l,m}(s, \bar{X}_{t_i} + (s - t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s - t_i) + \sigma\sqrt{s - t_i}w) \frac{d}{dw} p_{\mathcal{N}(0,1)}(w)\, dw \\
&= \sigma(s - t_i) \int_{\mathbb{R}} \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_{t_i} + (s-t_i)\bar{U}_{t_i}, \bar{U}_{t_i} + b(\bar{X}_{t_i}, \bar{U}_{t_i})(s-t_i) + \sigma\sqrt{s-t_i}w) p_{\mathcal{N}(0,1)}(w)\, dw
\end{aligned}
$$

The last equality is obtained from an integration by parts. By Lemma 4.6, we have that $\partial_x F$ is a bounded function, thus $\partial_x F_{k,l,m}$ is also bounded, and as $p_{\mathcal{N}(0,1)}(w) \to 0$ as $|w| \to \infty$, the boundary terms from the i.b.p. are 0.

We can rewrite:

$$
\mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (W_s - W_{t_i})\partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds = \sigma^2 \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (s - t_i) \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s)\, ds.
$$

Finally, we obtain that:

$$
\epsilon_{\mathrm{BR}}^{\bar{X}}(i) = \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (s - t_i) \left(b(\bar{X}_{t_i}, \bar{U}_{t_i}) \frac{\partial F_{k,l,m}}{\partial x}(s, \bar{X}_s, \bar{U}_s) + \sigma^2 \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s) \right) ds. \tag{3.22}
$$

The $(s - t_i)$ factor in the integral allows us to obtain the linear decrease of the error in Δt, so we express all the other error terms in this form. Similar calculations give:

$$
\epsilon_{\mathrm{NoR}}^{\bar{X}}(i) = \mathbb{E}\mathbb{1}_{\{\theta_i = t_i\}} \int_{t_i}^{t_{i+1}} (s - t_i) \left(b(\bar{X}_{t_i}, \bar{U}_{t_i}) \frac{\partial F_{k,l,m}}{\partial x}(s, \bar{X}_s, \bar{U}_s) + \sigma^2 \frac{\partial^2 F_{k,l,m}}{\partial u \partial x}(s, \bar{X}_s, \bar{U}_s) \right) ds. \tag{3.23}
$$

Contribution to the error after the reflection

We analyze now the contribution to the error produced by the discretisation of the drift in the position process, after reflection on any interval $[\theta_i, t_{i+1}]$, given by:

$$
\begin{aligned}
\epsilon_{\text{AR}}^{\bar{X}}(i) &= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} (\bar{U}_s + \bar{U}_{\eta(s)}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&= \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{\theta_i}^{t_{i+1}} \Big[- b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma(W_{\theta_i} - W_{t_i}) \\
&\qquad + b(\bar{X}_{\theta_i}, \bar{U}_{\theta_i})(s - \theta_i) + \sigma(W_s - W_{\theta_i}) \Big] \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds.
\end{aligned} \tag{3.24}
$$

The terms that involve Brownian increments are analysed separately starting with the increment before the jump, in the same way as the previous paragraph, in order to obtain a term of the type $w p_{\mathcal{N}(0,1)}(w)$:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} \partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma(W_s - W_{\theta_i}))\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} \mathbb{E}\big[\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, \bar{U}_{\theta_i} + b(0, \bar{U}_{\theta_i})(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i}\big] ds \Big| \mathcal{F}_{t_i}\right].
\end{aligned}
$$

In order to simplify notations, we introduce the function $I \colon \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$
I(u, \theta_i, s) = \mathbb{E}\big[\partial_x F_{k,l,m}(s, -(s-\theta_i)\bar{U}_{t_i}, u + b(0,u)(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i}\big].
$$

The previous equality then becomes:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} I(\bar{U}_{\theta_i}, \theta_i, s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[(W_{\theta_i} - W_{t_i}) \int_{\theta_i}^{t_{i+1}} I(-\bar{U}_{t_i} - b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma(W_{\theta_i} - W_{t_i}), \theta_i, s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma\mathbb{E}\left[\sqrt{\theta_i - t_i} \int_{\theta_i}^{t_{i+1}} ds \int_{\mathbb{R}} w I(-\bar{U}_{t_i} - b(\bar{X}_{t_i}, \bar{U}_{t_i})(\theta_i - t_i) - \sigma\sqrt{\theta_i - t_i}\, w, \theta_i, s) p_{\mathcal{N}(0,1)}(w)\, dw \Big| \mathcal{F}_{t_i}\right],
\end{aligned}
$$

and just as before, we can perform an integration by parts with $w p_{\mathcal{N}(0,1)}(w) = -p'_{\mathcal{N}(0,1)}(w)$ to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}} \sigma(W_{\theta_i} - W_{t_i}) \partial_x F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \Big| \mathcal{F}_{t_i}\right] \\
&= \sigma^2 \mathbb{E}\left[\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} (\theta_i - t_i) \int_{\theta_i}^{t_{i+1}} \int_{\mathbb{R}} \frac{\partial I}{\partial u}(\bar{U}_{\theta_i}, \theta_i, s) p_{\mathcal{N}(0,1)}(w)\, dw\right]
\end{aligned}
$$

where:

$$
\frac{\partial I}{\partial u}(u, \theta_i, s) = \mathbb{E}\bigg[\left(1 + (s-\theta_i)\frac{\partial b}{\partial u}(0,u)\right) \frac{\partial^2 F_{k,l,m}}{\partial x \partial u}(s, -(s-\theta_i)\bar{U}_{t_i}, u + b(0,u)(s-\theta_i) + \sigma(W_s - W_{\theta_i})) \mid \mathcal{F}_{\theta_i}\bigg]
$$

so by combining the different results:

$$\mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\sigma(W_{\theta_i}-W_{t_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\right]$$
$$=-\sigma^2\mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}(\theta_i-t_i)\int_{\theta_i}^{t_{i+1}}\left(1+(s-\theta_i)\frac{\partial b}{\partial u}(0,\bar{U}_{\theta_i})\right)\frac{\partial^2 F}{\partial x\partial u}(s,\bar{X}_s,\bar{U}_s)\,ds\right].$$

We now consider the case of the Brownian increment after the jump $(W_s-W_{\theta_i})$, which is independent from $\mathcal{F}_{\theta_i}$, so the calculations will be similar to those for $\epsilon^{\bar{X}}_{\text{BR}}(i)$:

$$\mathbb{E}\left[\int_{\theta_i}^{t_{i+1}}\sigma(W_s-W_{\theta_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\middle|\mathcal{F}_{\theta_i}\right]$$
$$=\sigma\int_{\theta_i}^{t_{i+1}}\mathbb{E}\big[(W_s-W_{\theta_i})\partial_x F_{k,l,m}(s,-(s-\theta_i)\bar{U}_{t_i},\bar{U}_{\theta_i}+b(0,\bar{U}_{\theta_i})(s-\theta_i)+\sigma(W_s-W_{\theta_i}))\mid\mathcal{F}_{\theta_i}\big]\,ds$$
$$=\sigma\int_{\theta_i}^{t_{i+1}}\sqrt{s-\theta_i}ds\int_{\mathbb{R}}w\partial_x F_{k,l,m}(s,-(s-\theta_i)\bar{U}_{t_i},\bar{U}_{\theta_i}+b(0,\bar{U}_{\theta_i})(s-\theta_i)+\sigma\sqrt{s-\theta_i}w)p_{\mathcal{N}(0,1)}(w)dw,$$

and after applying once more an i.b.p. (with null boundary terms since $\partial_x F_{k,l,m}$ is bounded and as $|u|\to+\infty$, $p_{\mathcal{N}(0,1)}(w)\to 0$) we obtain:

$$\mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\sigma(W_s-W_{\theta_i})\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)ds\right]$$
$$=\sigma^2\mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(s-\theta_i)\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)ds\right].$$

And by combining all these terms in (3.24), we obtain:

$$\begin{aligned}\epsilon^{\bar{X}}_{\text{AR}}(i)&=\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(\bar{U}_{\eta(s)}+\bar{U}_s)\partial_x F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\\&=-\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}(\theta_i-t_i)\int_{\theta_i}^{t_{i+1}}b(\bar{X}_{t_i},\bar{U}_{t_i})\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\\&\quad+\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}(\theta_i-t_i)\int_{\theta_i}^{t_{i+1}}\left(1+(s-\theta_i)\frac{\partial b}{\partial u}(0,\bar{U}_{\theta_i})\right)\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}(s,\bar{X}_s,\bar{U}_s)\,ds\\&\quad-\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}b(\bar{X}_{\theta_i},\bar{U}_{\theta_i})(s-\theta_i)\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\\&\quad+\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(s-\theta_i)\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds.\end{aligned}\tag{3.25}$$

Bounding the errors $\epsilon^{\bar{X}}$ of the position component

By using ($H_{Langevin}$)-(ii), we bound in (3.22):

$$
\begin{aligned}
|\epsilon^{\bar{X}}_{\mathrm{BR}}(i)| &\leq \left|\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}(s-t_i)b(\bar{X}_{t_i},\bar{U}_{t_i})\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\quad+\left|\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\sigma^2(s-t_i)\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\leq \Delta t\max\{\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})},\sigma^2\}\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\left(\left|\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\right|+\left|\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\right|\right)ds \\
&\leq \Delta t\max\{\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})},\sigma^2\}\mathbb{E}\int_{t_i}^{t_{i+1}}\left(\left|\frac{\partial F_{k,l,m}}{\partial x}(s,\bar{X}_s,\bar{U}_s)\right|+\left|\frac{\partial^2 F_{k,l,m}}{\partial u\partial x}(s,\bar{X}_s,\bar{U}_s)\right|\right)ds.
\end{aligned}
\tag{3.26}
$$

By (3.23), $\epsilon^{\bar{X}}_{\mathrm{NoR}}(i)$ is bounded by the same term. And in (3.25), by (H_{PDE})-(i):

$$
\begin{aligned}
|\epsilon^{\bar{X}}_{\mathrm{AR}}(i)| &\leq \Delta t\,\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\,\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\sigma^2\left(1+\Delta t\left\|\frac{\partial b}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\right)\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\,\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\,\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\quad+\Delta t\sigma^2\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds \\
&\leq \Delta t\left(C_b\mathbb{E}\int_{t_i}^{t_{i+1}}\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)\,ds+\Delta t C_{\partial_u b,\sigma,T}\mathbb{E}\int_{t_i}^{t_{i+1}}\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)ds\right)
\end{aligned}
\tag{3.27}
$$

where C_b is a constant that only depends on b and $C_{\partial_u b,\sigma,T}$ depends only on $\partial_u b$, σ, and T.

Combining these results and summing from $i=0$ to $i=N-1$, we obtain:

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\left(|\epsilon^{\bar{X}}_{\mathrm{BR}}(i)|+|\epsilon^{\bar{X}}_{\mathrm{AR}}(i)|+|\epsilon^{\bar{X}}_{\mathrm{NoR}}(i)|\right) \\
&\leq \Delta t\times C_{b,\partial_u b,\sigma,T}\left(\mathbb{E}\int_0^T\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s)\,ds+\mathbb{E}\int_0^T\left|\frac{\partial^2 F_{k,l,m}}{\partial x\partial u}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\right).
\end{aligned}
\tag{3.28}
$$

3.3 Analysis of the contribution to the error of the discretized drift on the velocity process

Error contribution before the reflection

We now consider the second term of (3.19), which represents the error introduced by the discretization of the drift of the velocity before the jump. Since $F_{k,l,m}$ is a smooth function, we apply Ito's formula. For the term corresponding to the contribution before the jump, we have:

$$
\begin{aligned}
\epsilon^{\bar{U}}_{\mathrm{BR}}(i) &= \mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}\left(b(\bar{X}_s,\bar{U}_s)-b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\right)\partial_u F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)ds\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{t_i}^{\theta_i}ds\int_{t_i}^{s}\left(\frac{\partial}{\partial t}+\mathcal{L}_{\mathrm{BR}}\right)\left(\left(b(\bar{X}_q,\bar{U}_q)-b(\bar{X}_{\nu(q)},\bar{U}_{\nu(q)})\right)\partial_u F_{k,l,m}(q,\bar{X}_q,\bar{U}_q)\right)dq\right].
\end{aligned}
$$

The local martingale that results from the application of Ito's formula is actually a true martingale by considering (H_{PDE})-(i) which gives that the drift b and its derivatives are uniformly bounded and $\partial_u F_{k,l,m}, \partial^2_{uu} F_{k,l,m} \in L^\infty(Q_T)$, for fixed $(k,l,m) \in \mathbb{N}^3$. By the definition for $F_{k,l,m}$ in (3.5) and the fact that $F \in L^\infty(Q_T)$we have that for any $(t,x,u) \in Q_T$

$$\left|\partial^2_{uu} F_{k,l,m}(t,x,u)\right| \leq \|F\|_{L^\infty(Q_T)} \int_{Q_T} \beta_k(t-\tau)\rho_l(x-y)\left|\partial^2_{uu} g_m(u-v)\right| d\tau dy dv \leq m^2 \|F\|_{L^\infty(Q_T)} .$$

Similarly, we show that $\partial_u F_{k,l,m}$ is also bounded for fixed $(k,l,m) \in \mathbb{N}^3$.

We distribute the linear differential operator (recall that the drift b does not depend on time) in the inner integral:

$$\begin{aligned}
I_s &:= \int_{t_i}^s \left(\frac{\partial}{\partial t} + \mathcal{L}_{\text{BR}}\right) \left(\left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \partial_u F_{k,l,m}(q, \bar X_q, \bar U_q)\right) dq \\
&= \int_{t_i}^s \left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \frac{\partial}{\partial u}\frac{\partial}{\partial t} F_{k,l,m}(q, \bar X_q, \bar U_q)\, dq \\
&\quad + \int_{t_i}^s \mathcal{L}_{\text{BR}} \left(\left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \frac{\partial}{\partial u} F_{k,l,m}(q, \bar X_q, \bar U_q)\right) dq \\
&= -\int_{t_i}^s \left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \frac{\partial}{\partial u} L F_{k,l,m}(q, \bar X_q, \bar U_q)\, dq \\
&\quad + \int_{t_i}^s \left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \partial_u R_{k,l,m}(q, \bar X_q, \bar U_q)\, dq \\
&\quad + \int_{t_i}^s \mathcal{L}_{\text{BR}} \left(\left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \frac{\partial}{\partial u} F_{k,l,m}(q, \bar X_q, \bar U_q)\right) dq
\end{aligned}$$

where we have used the fact that $\partial_t \partial_u F_{k,l,m} = -\partial_u L F_{k,l,m} + \partial_u R_{k,l,m}$ on Q_T in the last equality. Since

$$\mathcal{L}_{\text{BR}} \circ \partial_u = \partial_u \circ \mathcal{L}_{\text{BR}},$$

we obtain that :

$$\begin{aligned}
I_s &= \int_{t_i}^s \left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \frac{\partial}{\partial u} (\mathcal{L}_{\text{BR}} - L) F_{k,l,m}(q, \bar X_q, \bar U_q)\, dq \\
&\quad + \sigma^2 \int_{t_i}^s \partial_u b(\bar X_q, \bar U_q) \frac{\partial^2}{\partial u^2} F_{k,l,m}(q, \bar X_q, \bar U_q)\, dq \\
&\quad + \int_{t_i}^s \left[\left(\bar U_{\eta(q)} \partial_x + b(\bar X_{\nu(q)}, \bar U_{\nu(q)}) \partial_u + \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right) b(\bar X_q, \bar U_q)\right] \frac{\partial}{\partial u} F_{k,l,m}(q, \bar X_q, \bar U_q) dq \\
&\quad + \int_{t_i}^s \left(b(\bar X_q, \bar U_q) - b(\bar X_{\nu(q)}, \bar U_{\nu(q)})\right) \partial_u R_{k,l,m}(q, \bar X_q, \bar U_q) dq.
\end{aligned} \tag{3.29}$$

Coming back to the definition of $\mathcal{L}_{\text{BR}}$ and L we have

$$\begin{aligned}
&\partial_u (\mathcal{L}_{\text{BR}} - L) F_{k,l,m}(q, \bar X_q, \bar U_q) \\
&= \partial_u[-(\bar U_q - \bar U_{\nu(q)}) \partial_x F_{k,l,m}(q, \bar X_q, \bar U_q) - \Delta b_q \partial_u F_{k,l,m}(q, \bar X_q, \bar U_q)] \\
&= -\partial_x F_{k,l,m}(q, \bar X_q, \bar U_q) - \partial_u b(\bar X_q, \bar U_q) \partial_u F_{k,l,m}(q, \bar X_q, \bar U_q) \\
&\quad - (\bar U_q - \bar U_{\eta(q)}) \partial^2_{xu} F_{k,l,m}(q, \bar X_q, \bar U_q) - \Delta b_q \partial^2_{uu} F_{k,l,m}(q, \bar X_q, \bar U_q)]
\end{aligned}$$

We denote by $\Delta b_q = b(\bar{X}_q, \bar{U}_q) - b(\bar{X}_{\nu(q)}, \bar{U}_{\nu(q)})$ and the previous equality results in:

$$
\begin{aligned}
I_s = &- \int_{t_i}^{s} \Delta b_q \frac{\partial}{\partial x} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q) dq \\
&+ \int_{t_i}^{s} \Delta b_q \left(\bar{U}_{\eta(q)} - \bar{U}_q\right) \frac{\partial^2}{\partial x \partial u} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \left(\sigma^2 \partial_u b(\bar{X}_q, \bar{U}_q) - (\Delta b_q)^2\right) \frac{\partial^2}{\partial u^2} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \left[\left(\bar{U}_{\eta(q)} \partial_x + \left(b(\bar{X}_{\nu(q)}, \bar{U}_{\nu(q)}) - \Delta b_q\right) \partial_u + \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2}\right) b(\bar{X}_q, \bar{U}_q)\right] \frac{\partial}{\partial u} F_{k,l,m}(q, \bar{X}_q, \bar{U}_q)\, dq \\
&+ \int_{t_i}^{s} \Delta b_q \partial_u R_{k,l,m}(q, \bar{X}_q, \bar{U}_q) dq.
\end{aligned}
\tag{3.30}
$$

Finally, an i.b.p. is applied on $\int_{t_i}^{\theta_i} I_s ds$, by noting that $(\theta_i - s)' = -1$ in order to remove the inner integral:

$$
\begin{aligned}
\epsilon_{\text{BR}}^{\bar{U}}(i) = &- \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \Delta b_s \frac{\partial}{\partial x} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \Delta b_s \left(\bar{U}_{\eta(s)} - \bar{U}_s\right) \frac{\partial^2}{\partial x \partial u} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \left(\sigma^2 \partial_u b(\bar{X}_s, \bar{U}_s) - (\Delta b_s)^2\right) \frac{\partial^2}{\partial u^2} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \left[\left(\bar{U}_{\eta(s)} \partial_x + \left(b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)}) - \Delta b_s\right) \partial_u + \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2}\right) b(\bar{X}_s, \bar{U}_s)\right] \\
&\qquad \times \frac{\partial}{\partial u} F_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds \\
&+ \mathbb{E}\mathbb{1}_{\{\theta_i \in (t_i, t_{i+1})\}} \int_{t_i}^{\theta_i} (\theta_i - s) \Delta b_s \partial_u R_{k,l,m}(s, \bar{X}_s, \bar{U}_s) ds.
\end{aligned}
\tag{3.31}
$$

The term $\epsilon_{\text{NoR}}^{\bar{U}}(i)$ which corresponds to the error produced by the discretization of the drift of the velocity process in the case where no collision occurs, takes the same form as the previous formula, only requiring to replace θ_i by t_{i+1}.

Error contribution after the reflection

Similar computations to the previous paragraph are used to show that error introduced by the discretization of the drift of the velocity after the collision is:

$$
\begin{aligned}
&\epsilon_{\text{AR}}^{\bar U}(i) = -\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(\theta_i-s)\Delta b_s\frac{\partial}{\partial x}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\Delta b_s\left(\text{sign}(\bar Y_s)\bar U_{\eta(s)}-\bar U_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\left(\sigma^2\partial_u b(\bar X_s,\bar U_s)-(\Delta b_s)^2\right)\frac{\partial^2}{\partial u^2}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\left[\left(\text{sign}(\bar Y_s)\bar U_{\eta(s)}\partial_x+\left(b(\bar X_{\nu(s)},\bar U_{\nu(s)})-\Delta b_s\right)\partial_u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)b(\bar X_s,\bar U_s)\right]\\
&\qquad\times\frac{\partial}{\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&+\mathbb{E}\mathbb{1}_{\{\theta_i\in(t_i,t_{i+1})\}}\int_{\theta_i}^{t_{i+1}}(t_{i+1}-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds.
\end{aligned}
\tag{3.32}
$$

We proceed to regroup the errors in the drift of the velocity before and after the collision by introducing the following function $\nu^R\colon \mathbb{R}^+\mapsto\mathbb{R}^+$ defined as:

$$
\nu^R(t)=\begin{cases} t_{i+1} & \text{if } \theta_i=t_i\\ \theta_i & \text{if } t\in[t_i,\theta_i) \text{ and } \theta_i\in(t_i,t_{i+1})\\ t_{i+1} & \text{if } t\in[\theta_i,t_{i+1}) \text{ and } \theta_i\in(t_i,t_{i+1})\end{cases}
\tag{3.33}
$$

and summing up (3.31) and (3.32) on all intervals for $i=0$ to $N-1$:

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\left(\epsilon_{\text{BR}}^{\bar U}(i)+\epsilon_{\text{AR}}^{\bar U}(i)+\epsilon_{\text{NoR}}^{\bar U}(i)\right)=\mathbb{E}\int_0^T\left(b(\bar X_s,\bar U_s)-b(\bar X_{\nu(s)},\bar U_{\nu(s)})\right)\partial_u f(s,\bar X_s,\bar U_s)\,ds=\\
&=-\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\frac{\partial}{\partial x}F_{k,l,m}(s,\bar X_s,\bar U_s)ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\left(\text{sign}(\bar Y_s)\bar U_{\eta(s)}-\bar U_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\left(\sigma^2\partial_u b(\bar X_s,\bar U_s)-(\Delta b_s)^2\right)\frac{\partial^2}{\partial u^2}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\left[\left(\text{sign}(\bar Y_s)\bar U_{\eta(s)}\partial_x+\left(b(\bar X_{\nu(s)},\bar U_{\nu(s)})-\Delta b_s\right)\partial_u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)b(\bar X_s,\bar U_s)\right]\\
&\qquad\times\frac{\partial}{\partial u}F_{k,l,m}(s,\bar X_s,\bar U_s)\,ds\\
&\quad+\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds.
\end{aligned}
\tag{3.34}
$$

3.4 Bounds on the global error

To obtain the bounds on the error, we rely on theorem 2.1. In order to obtain L^2 norms, we integrate w.r.t. to the distribution of the discretised process. A simple case where this distribution is explicit is the one without drift on the velocity component, so we apply Girsanov's theorem to remove this drift. We introduce a new probability measure $\bar{\mathbb{Q}}$ defined using Girsanov's theorem:

$$\left.\frac{d\mathbb{P}}{d\bar{\mathbb{Q}}}\right|_{\mathcal{F}_T} = \mathcal{Z}_T = \exp\left(\int_0^T b(\bar{x}_s, \bar{u}_s)\, dW_s^0 - \frac{1}{2}\int_0^T b^2(\bar{x}_s, \bar{u}_s)\, ds\right),$$

where $(W_t^0)_{0<t<T}$ is a Brownian motion under $\bar{\mathbb{Q}}$. Since b is bounded, this means that the martingale $(\mathcal{Z}_t)_{0\leq t\leq T}$ admits moments of all orders.

We recall that for any $t \in [0,T]$, $|\nu^R(t) - t| \leq \Delta t$ and considering the first term of the equality (3.34), we have that:

$$\left|\mathbb{E}\int_0^T (\nu^R(s) - s)\Delta b_s \frac{\partial}{\partial x}F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)ds\right| \leq C_b \int_0^T \mathbb{E}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s, \bar{X}_s, \bar{U}_s)ds \times \Delta t\,. \quad (3.35)$$

And we have that

$$\begin{aligned}\mathbb{E}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s, \bar{X}_s, \bar{U}_s) &= \mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s\left|\frac{\partial}{\partial x}F_{k,l,m}\right|(s, \bar{x}_s, \bar{u}_s)\\ &\leq \left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s^2\right)^{\frac{1}{2}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial}{\partial x}F_{k,l,m}\right|^2(s, \bar{x}_s, \bar{u}_s)\right)^{\frac{1}{2}}\\ &\leq C_{\mu_0,b,\sigma,T}\left\|\frac{\partial}{\partial x}F_{k,l,m}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\end{aligned} \quad (3.36)$$

while for the second term of the equality (3.34) we have that

$$\begin{aligned}&\left|\mathbb{E}\int_0^T (\nu^R(s) - s)\left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\right)\left(\text{sign}(\bar{Y}_s)\bar{U}_{\eta(s)} - \bar{U}_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds\right|\\ &\leq C_b\int_0^T \mathbb{E}\left(|\bar{U}_{\eta(s)}| + |\bar{U}_s|\right)\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|(s, \bar{X}_s, \bar{U}_s)\, ds \times \Delta t\end{aligned} \quad (3.37)$$

where C_b depends only on the upper bound of the drift b. By choosing two positive numbers p, q such that $q > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality:

$$\begin{aligned}\mathbb{E}\,|\bar{U}_{\eta(s)}|\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|(s, \bar{X}_s, \bar{U}_s) &\leq \left(\mathbb{E}\,|\bar{U}_{\eta(s)}|^q\right)^{\frac{1}{q}}\left(\mathbb{E}\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^p(s, \bar{X}_s, \bar{U}_s)\right)^{\frac{1}{p}}\\ &\leq \left(\mathbb{E}_{\bar{\mathbb{Q}}}\bar{u}_{\eta(s)}^q\mathcal{Z}_{\eta(s)}\right)^{\frac{1}{q}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^p(s, \bar{X}_s, \bar{U}_s)\right)^{\frac{1}{p}}\\ &\leq \left(\mathbb{E}_{\bar{\mathbb{Q}}}\bar{u}_{\eta(s)}^{2q}\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_{\eta(s)}^2\right)^{\frac{1}{2q}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_s^{\frac{2}{2-p}}\right)^{\frac{2-p}{2p}}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right|^2(s, \bar{x}_s, \bar{u}_s)\right)^{\frac{1}{2}}\\ &\leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\end{aligned} \quad (3.38)$$

where $C_{\mu_0,b,\sigma,T,p,q}$ depends on the $2q-$moment of μ_0, the bound on b, the diffusion term σ, final time T, p and q. We perform the same calculation for the second term of (3.37) and obtain that:

$$\begin{aligned}&\left|\mathbb{E}\int_0^T (\nu^R(s) - s)\left(b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})\right)\left(\text{sign}(\bar{Y}_s)\bar{U}_{\eta(s)} - \bar{U}_s\right)\frac{\partial^2}{\partial x\partial u}F_{k,l,m}(s, \bar{X}_s, \bar{U}_s)\, ds\right|\\ &\leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F_{k,l,m}\right\|_{L^2(Q_T)} \times \Delta t \leq C_{\mu_0,b,\sigma,T,p,q}\left\|\frac{\partial^2}{\partial x\partial u}F\right\|_{L^2(Q_T)} \times \Delta t\end{aligned} \quad (3.39)$$

since $F_{k,l,m}$ is a convolution of F where $\partial_{xu}F \in L^2(Q_T)$ by Theorem 2.1.

Since $\Delta b_s = b(\bar{X}_s, \bar{U}_s) - b(\bar{X}_{\nu(s)}, \bar{U}_{\nu(s)})$, is such that $|\Delta b_s| \leq 2\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R})}$ so the third term of the inequality (3.34) is bounded by:

$$\begin{aligned}
&\left|\mathbb{E}\int_0^T (\nu^R(s)-s)\left(\sigma^2\frac{\partial b}{\partial u}(\bar{X}_s,\bar{U}_s) - (\Delta b_s)^2\right)\frac{\partial^2 F_{k,l,m}}{\partial u^2}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \leq \\
&\leq C_{b,\partial_u b,\sigma,T}\int_0^T \mathbb{E}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s)\,ds \times \Delta t
\end{aligned}$$

$C_{b,\partial_u b,\sigma,T}$ depends only on the L^∞ norm of b and $\partial_u b$ and on σ. By Girsanov's theorem:

$$\begin{aligned}
\mathbb{E}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s) &= \mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_T\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{x}_s,\bar{u}_s) \leq \left(\mathbb{E}_{\bar{\mathbb{Q}}}\mathcal{Z}_T^2\right)^{\frac12}\left(\mathbb{E}_{\bar{\mathbb{Q}}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\bar{x}_s,\bar{u}_s)\right)^{\frac12} \\
&\leq \exp\left(\frac34\|b\|^2_{L^\infty}T\right)\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\bar{p}^c(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac12} \\
&\leq \exp\left(\frac34\|b\|^2_{L^\infty}T\right)\|\bar{p}^c(s;\cdot,\cdot)\|^{\frac12}_{L^\infty(\mathcal{D}\times\mathbb{R})}\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac12} \\
&\leq \exp\left(\frac34\|b\|^2_{L^\infty}T\right)\|2\mu_0\|^{\frac12}_{L^\infty(\mathcal{D}\times\mathbb{R})}\left(\int_{\mathcal{D}\times\mathbb{R}}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|^2(s,\xi,\zeta)\,d\xi d\zeta\right)^{\frac12},
\end{aligned} \tag{3.40}$$

where μ_0 is the p.d.f. of the initial values so it follows ($H_{\textit{Weak Error}}$)-(i), meaning that $\mu_0 \in L^\infty(\mathcal{D}\times\mathbb{R})$. Integrated on $[0,T]$, we get that:

$$\begin{aligned}
&\int_0^T \mathbb{E}\left|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right|(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\leq e^{\left(\frac32\|b\|^2_{L^\infty(\mathcal{D}\times\mathbb{R})}T\right)}\|2\mu_0\|^{\frac12}_{L^\infty(\mathcal{D}\times\mathbb{R})}\int_0^T\left\|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}ds \\
&\leq \sqrt{T}e^{\left(\frac32\|b\|^2_{L^\infty(\mathcal{D}\times\mathbb{R})}T\right)}\|\mu_0\|^{\frac12}_{L^\infty(\mathcal{D}\times\mathbb{R})}\left\|\frac{\partial^2 F_{k,l,m}}{\partial u^2}\right\|_{L^2(Q_T)} \leq \sqrt{T}e^{\left(\frac32\|b\|^2_{L^\infty(\mathcal{D}\times\mathbb{R})}T\right)}\|\mu_0\|^{\frac12}_{L^\infty(\mathcal{D}\times\mathbb{R})}\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)}.
\end{aligned}$$

Concerning the third term of (3.34):

$$\begin{aligned}
&\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\left[\left(\mathrm{sign}(\bar{Y}_s)\bar{U}_{\eta(s)}\partial_x + b(\bar{X}_{\nu(s)},\bar{U}_{\nu(s)})\partial_u + \frac{\sigma^2}{2}\partial_{uu}\right)b(\bar{X}_s,\bar{U}_s)\right]\frac{\partial}{\partial u}F_{k,l,m}(s,\bar{X}_s,\bar{U}_s)\,ds\right| \\
&\leq C_{b,\partial_x b,\partial_u b,\sigma,T}\int_0^T\mathbb{E}\left(1+|\bar{U}_{\eta(s)}|\right)\left|\frac{\partial F_{k,l,m}}{\partial u}\right|(s,\bar{X}_s,\bar{U}_s)\,ds\times\Delta t \\
&\leq C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}\left\|\frac{\partial F_{k,l,m}}{\partial u}\right\|_{L^2(Q_T)}\times\Delta t = C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}\times\Delta t\,,
\end{aligned} \tag{3.41}$$

where $C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T,p,q}$ depends on the $2q-$moment of μ_0, the bound on b and its derivatives, on σ, final time T, on p and q.

Regarding the last term of (3.34), we use the expression of the error written in Corollary 3.1:

$$\left|\mathbb{E}\int_0^T(\nu^R(s)-s)\Delta b_s\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\,ds\right| \leq C_b\mathbb{E}\int_0^T\left|\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right|\,ds\times\Delta t \tag{3.42}$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left|\partial_u R_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right| ds \leq \mathbb{E}\int_0^T \left|\partial_u R^{\mathrm{Sp}}_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right| ds + \mathbb{E}\int_0^T \left|\partial_u R^{\mathrm{Tm}}_{k,l,m}[F](s,\bar{X}_s,\bar{U}_s)\right| ds \\
&\leq \mathbb{E}\int_0^T \left|\left(\frac{\partial^2}{\partial u \partial x} F * ((u g_m)\rho_l \beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds + \mathbb{E}\int_0^T \left|\frac{\partial}{\partial u} b(\bar{X}_s,\bar{U}_s)\left(\frac{\partial}{\partial u} F * (g_m \rho_l \beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds \\
&\quad + \mathbb{E}\int_0^T \left|\left(\left(\frac{\partial}{\partial u}(bF)\right) * (g_m \rho_l \beta_k)\right)\right|(s,\bar{X}_s,\bar{U}_s)\,ds + \mathbb{E}\int_0^T \beta_k(s)\left|\left(\left(\frac{\partial}{\partial u} F(0,\cdot,\cdot)\right) * (g_m \rho_l)\right)\right|(\bar{X}_s,\bar{U}_s)\,ds \\
&\leq C_{\mu_0,b,\sigma,T}\left\|\frac{\partial^2 F}{\partial u \partial x}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\partial_u b,\sigma,T}\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\sigma,T}\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)} + C_{\mu_0,b,\sigma,T}\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}.
\end{aligned}
\tag{3.43}
$$

Combining all these terms gives us that:

$$
\begin{aligned}
&\left|\sum_{i=0}^{N-1}\left(\epsilon^{\bar{U}}_{\mathrm{BR}}(i) + \epsilon^{\bar{U}}_{\mathrm{AR}}(i) + \epsilon^{\bar{U}}_{\mathrm{NoR}}(i)\right)\right| \\
&\quad \leq C_{\mu_0,b,\partial_x b,\partial_u b,\sigma,T} \times \Delta t \\
&\qquad \times \left(\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)} + \left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)} + \left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(\mathcal{D}\times\mathbb{R})} + \left\|\frac{\partial^2 F}{\partial u \partial x}\right\|_{L^2(Q_T)} + \left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\right)
\end{aligned}
\tag{3.44}
$$

By the same technique we can show that (3.28) is bounded by

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\left(|\epsilon^{\bar{X}}_{\mathrm{BR}}(i)| + |\epsilon^{\bar{X}}_{\mathrm{AR}}(i)| + |\epsilon^{\bar{X}}_{\mathrm{NoR}}(i)|\right) \\
&\leq C_{b,\partial_u b,\sigma,T}\left(\mathbb{E}\int_0^T\left(\left|\frac{\partial F_{k,l,m}}{\partial x}\right|(s,\bar{X}_s,\bar{U}_s) + \left|\frac{\partial^2 F_{k,l,m}}{\partial x \partial u}\right|(s,\bar{X}_s,\bar{U}_s)\right) ds\right) \times \Delta t \\
&\leq C_{\mu_0,b,\partial_u b,\sigma,T}\left(\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)} + \left\|\frac{\partial^2 F}{\partial u \partial x}\right\|_{L^2(Q_T)}\right) \times \Delta t.
\end{aligned}
\tag{3.45}
$$

Going back to equality (3.17) and putting together all the various results:

$$
\begin{aligned}
&\left|\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0})-\mathbb{E}\psi(\bar X_T^{X_0,U_0},\bar U_T^{X_0,U_0})\right| = \left|\sum_{i=0}^{N-1}\left(\epsilon_{\text{BR}}(i)+\epsilon_{\text{AR}}(i)+\epsilon_{\text{NoR}}(i)\right)+\mathbb{E}F(0,X_0,U_0)+\epsilon^{\text{Reg}}_{k,l,m}\right|\\
&=\Bigg|\sum_{i=0}^{N-1}\left(\epsilon^{\bar X}_{\text{BR}}(i)+\epsilon^{\bar U}_{\text{BR}}(i)+\epsilon^{\bar X}_{\text{AR}}(i)+\epsilon^{\bar U}_{\text{AR}}(i)+\epsilon^{\bar X}_{\text{NoR}}(i)+\epsilon^{\bar U}_{\text{NoR}}(i)\right)+\epsilon^{\text{Reg}}_{k,l,m}\\
&\qquad-\mathbb{E}\int_0^T R^{\text{Sp}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds+\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t} R^{\text{Tm}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\Bigg|\\
&\le\sum_{i=0}^{N-1}\left(|\epsilon^{\bar X}_{\text{BR}}(i)|+|\epsilon^{\bar X}_{\text{AR}}(i)|+|\epsilon^{\bar X}_{\text{NoR}}(i)|\right)+\left|\sum_{i=0}^{N-1}\left(\epsilon^{\bar U}_{\text{BR}}(i)+\epsilon^{\bar U}_{\text{AR}}(i)+\epsilon^{\bar U}_{\text{NoR}}(i)\right)\right|+\left|\epsilon^{\text{Reg}}_{k,l,m}\right|\\
&\quad+\left|\mathbb{E}\int_0^T R^{\text{Sp}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t} R^{\text{Tm}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|\\
&\quad\le C_{\sigma,b,\partial_x b,\partial_u b,\mu_0,T}\Bigg(\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)}\\
&\qquad\qquad+\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^2(\mathcal{D}\times\mathbb{R})}\Bigg)\times\Delta t\\
&\quad+\left|\epsilon^{\text{Reg}}_{k,l,m}\right|+\left|\mathbb{E}\int_0^T R^{\text{Sp}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t} R^{\text{Tm}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|.
\end{aligned}
\tag{3.46}
$$

By Lemma 3.3 term $\left|\epsilon^{\text{Reg}}_{k,l,m}\right|+\left|\mathbb{E}F(0,X_0,U_0)-\mathbb{E}\int_0^{\Delta t} R^{\text{Tm}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|$ goes to zero as (k,l,m) go to infinity. By Lemma 5.3, the term $R^{\text{Sp}}_{k,l,m}[F]$ converges uniformly towards 0 as (k,l,m) go to infinity, if $l=m$, thus the term $\left|\mathbb{E}\int_0^T R^{\text{Sp}}_{k,l,m}[F](s,\bar X_s,\bar U_s)\,ds\right|$ also converges to 0.

We can therefore conclude that by taking $l=m$ and $(k,l,m)\to\infty$ in the inequality (3.46) we obtain that

$$
\begin{aligned}
&\left|\mathbb{E}\psi(X_T^{X_0,U_0},U_T^{X_0,U_0})-\mathbb{E}\psi(\bar X_T^{X_0,U_0},\bar U_T^{X_0,U_0})\right|\\
&\le\Delta t\times C_{\sigma,b,\partial_x b,\partial_u b,\mu_0,T}\\
&\quad\times\left(\left\|\frac{\partial F}{\partial u}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F}{\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u^2}\right\|_{L^2(Q_T)}+\left\|\frac{\partial^2 F}{\partial u\partial x}\right\|_{L^2(Q_T)}+\left\|\frac{\partial F(0,\cdot,\cdot)}{\partial u}\right\|_{L^\infty(\mathcal{D}\times\mathbb{R})}\right).
\end{aligned}
\tag{3.47}
$$

This ends the proof of Theorem 1.6: the weak error of our scheme converges at least linearly in the time discretization step Δt.

4 Regularity of the flow of the free Langevin process

In this section we prove the regularity result up to the first order of the F function stated in Theorem 2.1. The results are stated in Lemma 4.3 and Lemma 4.6.

They are based on the study of the regularity of the flow in sens of Bouleau and Hirsh, for the free Lagrangian process first, for it confined version then.

We consider the free Langevin process $(Y_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^d$ which verifies the equation:

$$\begin{cases} Y_t = x + \displaystyle\int_0^t V_s^{x,u}\, ds, \\ V_t = u + \sigma \widetilde{W}_t + \displaystyle\int_0^t \widetilde{b}(Y_s^{x,u}, V_s^{x,u})\, ds, \end{cases} \tag{4.1}$$

where $(x, u) \in \mathcal{D} \times \mathbb{R}^d$ and $\widetilde{b}\colon \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ defined as:

$$\widetilde{b}(x, u) := (b', \operatorname{sign}(x^{(d)}) b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right). \tag{4.2}$$

We recall the following notation: for any $x \in \mathbb{R}^d$, we write $x = (x', x^{(d)})$ where x' are the first $(d-1)$ coordinates of x and $x^{(d)}$ the d$^{\text{th}}$ component.

The result in [Bouleau and Hirsch, 1989] shows that the process $(Y_t^{x,u}, V_t^{x,u})_{t \geq 0}$ admits a derivative in the sense of distributions w.r.t. the initial conditions (x, u). This result allows us to state that the gradients $\nabla_x F$ and $\nabla_u F$ in Theorem 2.1 are well defined . We reproduce their technique and arguments in this section. It involves an augmentation of the probability space to include the initial conditions and a modified SDE on the new probability space. The modified SDE respects a weaker uniqueness condition which allows to perform some operations that are not allowed on the original SDE (4.1).

4.1 Derivability of the flow in the sens of Bouleau and Hirsch

We recall the notations and results of Bouleau and Hirsch in [Bouleau and Hirsch, 1989] for a general process $(X_t)_{0 \leq t \leq T}$ that is a solution of the stochastic differential equation:

$$X_t^x = x + \int_0^t b(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{4.3}$$

where the functions b and σ are Lipschitz with, at most, linear increase. Let $\Omega = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$, the Wiener space of continuous functions ω such that $\omega(0) = 0$ equipped with the metric of the uniform convergence on compacts. $\mathcal{F}$ is the Borel $\sigma-$algebra over Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. The canonical process is defined as $W_t(\omega) = \omega(t)$ for all $t \geq 0$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_t)$ is a Brownian motion. The authors enlarge the probability space as $\widetilde{\Omega} = \mathbb{R}^d \times \Omega$ and $\widetilde{\mathcal{F}}$ the Borel $\sigma-$algebra over $\widetilde{\Omega}$. $\widetilde{\mathbb{P}}$ is the product measure $h dx \otimes \mathbb{P}$ where h is a probability density that has a second order moment. The canonical process is therefore $\widetilde{W_t}(x, \omega) = W_t$ with natural filtration $\widetilde{\mathcal{F}}_t$ which is augmented by the $\widetilde{P}-$negligible sets of $\widetilde{\mathcal{F}}$. Then $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, \widetilde{\mathbb{P}}, \widetilde{W_t})$ is the canonical Brownian motion starting from 0. Let $e_1, \ldots, e_n$ be the canonical basis in $\mathbb{R}^d$. For every i in $\{1, \ldots, d\}$, the Dirichlet space $\widetilde{D}_i$ is defined as:

$$\widetilde{D}_i = \left\{ \begin{matrix} u\colon \widetilde{\Omega} \mapsto \mathbb{R},\ \exists \widetilde{u}\colon \widetilde{\Omega} \mapsto \mathbb{R} \text{ Borel measurable s.t. } u = \widetilde{u}, \widetilde{\mathbb{P}} - \text{a.e. and} \\ \forall (x, \omega) \in \widetilde{\Omega}, t \mapsto \widetilde{u}(x + te_i, \omega) \text{ is locally absolutely continuous} \end{matrix} \right\}$$

so $\widetilde{D}_i$ can be considered as a set of classes w.r.t. $\widetilde{\mathbb{P}}-$a.e. equality. If u is in $\widetilde{D}_i$ and $\widetilde{u}$ is associated with it according to the above definition, then:

$$\nabla_i u(x, \omega) = \varliminf_{t \to 0} \frac{\widetilde{u}(x + te_i, \omega) - \widetilde{u}(x, \omega)}{t}.$$

Let $\widetilde{D}$ be the Dirichlet space defined as:

$$\widetilde{D} = \left\{ u \in L^2(\widetilde{\mathbb{P}}) \bigcap \left(\bigcap_{i=1}^{d} \widetilde{D}_i \right); \forall 1 \leq i \leq d, \nabla_i u \in L^2(\widetilde{\mathbb{P}}) \right\}$$

equipped with the norm:

$$\|u\|_{\widetilde{D}} = \left(\int_{\widetilde{\Omega}} u^2 \, d\widetilde{\mathbb{P}} + \sum_{i=1}^{d} \int_{\widetilde{\Omega}} (\nabla_i u)^2 \, d\widetilde{\mathbb{P}} \right)^{\frac{1}{2}}.$$

We also consider the space $D = \{f \in L^2(hdx); \forall 1 \leq j \leq d \frac{\partial}{\partial x_j} f \in L^2(hdx)\}$ equipped with its usual norm. We introduce the process $(\widetilde{X}_t^x)_{0 \leq t \leq T}$ defined on the space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, \widetilde{\mathbb{P}}, \widetilde{W}_t)$ that solves the stochastic differential equation:

$$\widetilde{X}_t^x = x + \int_0^t b(\widetilde{X}_s^x) \, ds + \int_0^t \sigma(\widetilde{X}_s^x) \, d\widetilde{W}_s. \tag{4.4}$$

It can be shown that for every $0 \leq t \leq T$, $\widetilde{X}_t = X_t^x$, $\widetilde{\mathbb{P}}-$almost surely.

Theorem 4.1 ([Bouleau and Hirsch, 1989])**.**

(i) For $\mathbb{P}-$almost every ω, for all $0 \leq t \leq T$, $X_t^{\cdot}(\omega) \in D^d \subset (H^1_{loc}(\mathbb{R}^d))^d$

(ii) There exists a $(\widetilde{\mathcal{F}}_t)-$adapted $GL_d(\mathbb{R})-$valued continuous process $(M_t)_{0 \leq t \leq T}$ such that, for $\widetilde{\mathbb{P}}-$almost every ω,

$$\forall t \leq T \frac{\partial}{\partial x} (X_t^x(\omega)) = M_t(x, \omega) \quad dx - a.e.$$

where $\frac{\partial}{\partial x}$ denotes the derivative in the distribution sense.

And also:

Lemma 4.2. *$(M_t)_{0 \leq t \leq T}$ is the $\mathbb{R}^{d \times d}-$values $(\widetilde{\mathcal{F}}_t)-$ adapted continuous solution of the linear sde:*

$$M_t^i = e_i + \int_0^t b_x(\widetilde{X}_s^x) M_s^i \, ds + \sum_{j=1}^{d} \int_0^t \sigma_x^j(\widetilde{X}_s^x) M_s^i \, d\widetilde{W}_s^j$$

for all $1 \leq i \leq d$, where b_x and σ_x^j are versions of the almost everywhere derivatives of b and σ^j.

4.2 Application to the free Langevin process

We apply theorem 4.1 and lemma 4.2 to the process (4.1), since the function $\widetilde{b}$ is Lipschitz with linear growth and σ is a constant. Then there exists $(\widetilde{\mathcal{F}}_t)-$adapted processes, parametrised by $(x, u) \in \mathbb{R}^d \times \mathbb{R}^d$, $(M_t^Y(x, u))$, $(M_t^V(x, u))$, $(N_t^Y(x, u))$, $(N_t^V(x, u))$ such that:

$$\begin{cases} \nabla_x Y_t^{x,u} = M_t^Y(x, u) \\ \nabla_x V_t^{x,u} = M_t^V(x, u) \\ \nabla_u Y_t^{x,u} = N_t^Y(x, u) \\ \nabla_u V_t^{x,u} = N_t^V(x, u) \end{cases} \tag{4.5}$$

where

$$\begin{cases} M_t^Y(x,u) = I_d + \displaystyle\int_0^t M_s^V(x,u)\,ds \\ M_t^V(x,u) = \displaystyle\int_0^t \widetilde{b}_x(\widetilde{Y}_s^{x,u}, \widetilde{V}_s^{x,u}) M_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{Y}_s^{x,u}, \widetilde{V}_s^{x,u}) M_s^V(x,u)\,ds \\ N_t^Y(x,u) = \displaystyle\int_0^t N_s^V(x,u)\,ds \\ N_t^V(x,u) = I_d + \displaystyle\int_0^t \widetilde{b}_x(\widetilde{Y}_s^{x,u}, \widetilde{V}_s^{x,u}) N_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{Y}_s^{x,u}, \widetilde{V}_s^{x,u}) N_s^V(x,u)\,ds \end{cases} \tag{4.6}$$

where $\widetilde{b}_x$ and $\widetilde{b}_u$ are versions of the almost everywhere derivatives in x and u of $\widetilde{b}$. I_d is the identity in dimension d. Since

$$\widetilde{b}(x,u) = (b', \operatorname{sign}(x^{(d)}) b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right)$$

we take $\widetilde{b}_x(x,u) = (\nabla_{x'}\widetilde{b}, \partial_{x^{(d)}}\widetilde{b})(x,u)$ and $\widetilde{b}_u(x,u) = (\nabla_{u'}\widetilde{b}, \partial_{u^{(d)}}\widetilde{b})(x,u)$, where

$$\begin{cases} \nabla_{x'}\widetilde{b}(x,u) = (\nabla_{x'} b', \operatorname{sign}(x^{(d)}) \nabla_{x'} b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right) \\ \partial_{x^{(d)}}\widetilde{b}(x,u) = (\operatorname{sign}(x^{(d)}) \partial_{x^{(d)}} b', \partial_{x^{(d)}} b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)} u^{(d)})) \right) \\ \nabla_{u'}\widetilde{b}(x,u) = (\nabla_{u'} b', \operatorname{sign}(x^{(d)}) \nabla_{u'} b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right) \\ \partial_{u^{(d)}}\widetilde{b}(x,u) = (\operatorname{sign}(x^{(d)}) \partial_{u^{(d)}} b', \partial_{u^{(d)}} b^{(d)}) \left((x', |x^{(d)}|), (u', \operatorname{sign}(x^{(d)}) u^{(d)}) \right). \end{cases} \tag{4.7}$$

Properties of the weak derivatives in a no-drift setting

Let $(z_t^{x,u}, \eta_t^{x,u})_{0\le t\le T}$ be the process that solves the following SDE under a new probability measure $\mathbb{P}_{z,\eta}$:

$$\begin{cases} z_t^{x,u} = x + \displaystyle\int_0^t \eta_s^u\,ds \\ \eta_t^u = u + \sigma \check{W}_t \end{cases} \tag{4.8}$$

where $(\check{W}_t)_{0\le t\le T}$ is a Brownian motion under the new probability. We also consider the following processes defined by the equations:

$$\begin{cases} \check{M}_t^Y(x,u) = I_d + \displaystyle\int_0^t \check{M}_s^V(x,u)\,ds \\ \check{M}_t^V(x,u) = \displaystyle\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) \check{M}_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) \check{M}_s^V(x,u)\,ds \\ \check{N}_t^Y(x,u) = \displaystyle\int_0^t \check{N}_s^V(x,u)\,ds \\ \check{N}_t^V(x,u) = I_d + \displaystyle\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) \check{N}_s^Y(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u}, \widetilde{\eta}_s^u) \check{N}_s^V(x,u)\,ds. \end{cases} \tag{4.9}$$

We analyse the continuity at the boundary $\partial\mathcal{D}$ of the solutions of (4.9) starting with the term $\check{M}_t^V$.

Lemma 4.3. *For any $(t,u) \in [0,T] \times \mathbb{R}^d$ and $p \in [1,\infty)$, the processes $\check{M}_t^Y(\cdot,u)$, $\check{M}_t^V(\cdot,u)$, $\check{N}_t^Y(\cdot,u)$ and $\check{N}_t^V(\cdot,u)$ are continuous up to the boundary $\partial\mathcal{D}$ in norm L^p.*

Proof. This result is proved using Gronwall's lemma. The regularity of the derivatives $\widetilde{b}_x$ and $\widetilde{b}_u$ is used. The regularity of the density of the drift-less free Langevin model is used to smooth out the changes of sign when the boundary is hit.

Let $(t,x,u) \in Q_T$ and $\bar{x} \in \partial\mathcal{D}$ the projection of x on $\partial\mathcal{D}$. By the system (4.9):

$$\begin{aligned}
\left|\check{M}^V_t(x,u) - \check{M}^V_t(\bar{x},u)\right| &= \bigg|\int_0^t \widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(x,u)\,ds + \int_0^t \widetilde{b}_u(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\check{M}^V_s(x,u)\,ds \\
&\qquad - \int_0^t \widetilde{b}_x(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(\bar{x},u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\check{M}^V_s(\bar{x},u)\,ds\bigg| \\
&\le \left|\int_0^t \widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(\bar{x},u)\,ds\right| + \\
&\qquad + \left|\int_0^t \widetilde{b}_u(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\check{M}^V_s(x,u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\check{M}^V_s(\bar{x},u)\,ds\right|.
\end{aligned} \tag{4.10}$$

The first term in this sum corresponds to the first derivative in x while the second term of the sum corresponds to the first derivative in u. By (H_{PDE})-(i) and Gronwall's lemma it is easy to notice that there exists a constant $C_{\nabla_x b,\nabla_u b,T}$ such that:

$$\sup_{(t,x,u)\in\overline{Q_T}} \left(\left\|\check{M}^Y_t(x,u)\right\| + \left\|\check{M}^V_t(x,u)\right\| + \left\|\check{N}^Y_t(x,u)\right\| + \left\|\check{N}^V_t(x,u)\right\|\right) < C_{\nabla_x b,\nabla_u b,T}$$

so:

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\check{M}^Y_s(\bar{x},u)\,ds\right| \\
&\le \left|\int_0^t \widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s)\left(\check{M}^Y_s(x,u) - \check{M}^Y_s(\bar{x},u)\right)\,ds\right| \\
&\qquad + \left|\int_0^t \left(\widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s) - \widetilde{b}_x(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\right)\check{M}^Y_s(\bar{x},u)\,ds\right| \\
&\le \left\|\widetilde{b}_x\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\check{M}^Y_s(x,u) - \check{M}^Y_s(\bar{x},u)\right|\,ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}^Y_t(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\widetilde{b}_x(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s) - \widetilde{b}_x(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\right|\,ds \\
&\le \left\|\widetilde{b}_x\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t\int_0^s \left|\check{M}^V_\theta(x,u) - \check{M}^V_\theta(\bar{x},u)\right|\,d\theta ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}^Y_t(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s) - \nabla_{x'}\widetilde{b}(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\right|\,ds \\
&\qquad + \sup_{t\in[0,T]}\left\|\check{M}^Y_t(\bar{x},u)\right\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\int_0^t \left|\partial_{x^{(d)}}\widetilde{b}(\widetilde{z}^{x,u}_s,\widetilde{\eta}^u_s) - \partial_{x^{(d)}}\widetilde{b}(\widetilde{z}^{\bar{x},u}_s,\widetilde{\eta}^u_s)\right|\,ds.
\end{aligned} \tag{4.11}$$

The second term of this inequality represents the derivatives of the drift with respect to the first $d-1$ coordinates while the third term corresponds to the derivative w.r.t the d^{th} coordinate. These two terms are analyzed separately in the following paragraphs: **The derivative on the first *d-1* directions** and **The derivative on the *d*[th] direction**.

The derivative on the first *d-1* directions

Going back to the choices for the derivatives of $\widetilde{b}$ in (4.7), we have that:

$$\begin{aligned}
&\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \nabla_{x'}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right| ds \\
&\le \int_0^t \left|\nabla_{x'}b'\left(\left((\widetilde{z}_s^{x,u})',\left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right. \\
&\qquad \left. - \nabla_{x'}b'\left(\left((\widetilde{z}_s^{\bar{x},u})',\left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right| ds \\
&\quad + \int_0^t \left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{x,u})',\left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right. \\
&\qquad \left. - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})',\left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right| ds.
\end{aligned} \tag{4.12}$$

We recall that the derivative w.r.t. x of the drift b is Lipschitz continuous which we denote as $L_{\nabla_x b}$ its Lipschitz constant. Also the $d-$dimensional free Langevin process with no drift defined in (4.8) can be considered as being d independent $1-$dimensional free Langevin processes. This results in:

$$\begin{aligned}
&\int_0^t \left|\nabla_{x'}b'\left(\left((\widetilde{z}_s^{x,u})',\left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right. \\
&\quad \left. - \nabla_{x'}b'\left(\left((\widetilde{z}_s^{\bar{x},u})',\left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right| ds \\
&\le L_{\nabla_x b}\left(\int_0^t \left|\left|(\widetilde{z}_s^{x,u})^{(d)}\right| - \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right| ds + \int_0^t \left|(\widetilde{\eta}_s^u)^{(d)}\right|\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\right) \\
&\le L_{\nabla_x b}\left(\int_0^t \left|(\widetilde{z}_s^{x,u})^{(d)} - (\widetilde{z}_s^{\bar{x},u})^{(d)}\right| ds + \int_0^t \left|(\widetilde{\eta}_s^u)^{(d)}\right|\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\right).
\end{aligned} \tag{4.13}$$

For the second integral of (4.12), we have by the boundedness of $\nabla_x b$:

$$\begin{aligned}
&\int_0^t \left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{x,u})',\left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right. \\
&\qquad \left. - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})',\left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right| ds \\
&\le \|\nabla_x b\|_{L^\infty(Q_T,\mathbb{R}^{2d})}\int_0^t \left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds \\
&\quad + \int_0^t \left|\nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{x,u})',\left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right. \\
&\qquad \left. - \nabla_{x'}b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})',\left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right),\left((\widetilde{\eta}_s^u)',\mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\right| ds \\
&\le C_{\nabla_x b}\int_0^t \left(1 + \left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + L_{\nabla_x b}\int_0^t \left|(\widetilde{z}_s^{x,u})^{(d)} - (\widetilde{z}_s^{\bar{x},u})^{(d)}\right| ds
\end{aligned} \tag{4.14}$$

where $C_{\nabla_x b} = \max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\}$. Combining these two previous inequalities and using the definition of the free Langevin model with no drift (4.8), we go back to inequality (4.12) to obtain:

$$\begin{aligned}
&\int_0^t \left|\nabla_{x'}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \nabla_{x'}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right| ds \\
&\le C_{\nabla_x b}\left(\int_0^t \left(1 + \left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + t\,|x - \bar{x}|\right)
\end{aligned} \tag{4.15}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$.

The derivative on the d^{th} direction

We develop the third term of the inequality (4.11) based on the same arguments used in the previous section:

$$\begin{aligned}
&\int_0^t \left|\partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u) - \partial_{x^{(d)}}\widetilde{b}(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\right| ds \\
&\le \int_0^t \Big|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)\partial_{x^{(d)}}b'\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\partial_{x^{(d)}}b'\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| ds \\
&\quad + \int_0^t \Big|\partial_{x^{(d)}}b^{(d)}\left(\left((\widetilde{z}_s^{x,u})', \left|(\widetilde{z}_s^{x,u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right) \\
&\qquad - \partial_{x^{(d)}}b^{(d)}\left(\left((\widetilde{z}_s^{\bar{x},u})', \left|(\widetilde{z}_s^{\bar{x},u})^{(d)}\right|\right), \left((\widetilde{\eta}_s^u)', \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)(\widetilde{\eta}_s^u)^{(d)}\right)\right)\Big| ds \\
&\le C_{\nabla_x b}\left(\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds + t\,|x-\bar{x}|\right)
\end{aligned} \tag{4.16}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$. Finally, from inequality (4.11):

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_x(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^Y(x,u)\,ds - \int_0^t \widetilde{b}_x(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^Y(\bar{x},u)\,ds\right| \\
&\le C_{\nabla_x b}\left(T\int_0^t \left|\check{M}_\theta^V(x,u) - \check{M}_\theta^V(\bar{x},u)\right| d\theta + \int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\right) \\
&\quad + C_{\nabla_x b}t\,|x-\bar{x}|
\end{aligned} \tag{4.17}$$

where $C_{\nabla_x b} = 2\max\{L_{\nabla_x b}, \|\nabla_x b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\}$.

Similar calculations show that the second term from (4.10) is bounded by:

$$\begin{aligned}
&\left|\int_0^t \widetilde{b}_u(\widetilde{z}_s^{x,u},\widetilde{\eta}_s^u)\check{M}_s^V(x,u)\,ds - \int_0^t \widetilde{b}_u(\widetilde{z}_s^{\bar{x},u},\widetilde{\eta}_s^u)\check{M}_s^V(\bar{x},u)\,ds\right| \\
&\le \|\nabla_u b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d}}\int_0^t \left|\check{M}_s^V(x,u) - \check{M}_s^V(\bar{x},u)\right| ds + C_{\nabla_u b}t\,|x-\bar{x}| \\
&\quad + C_{\nabla_u b}\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds
\end{aligned} \tag{4.18}$$

where $C_{\nabla_u b} = 2\max\{L_{\nabla_u b}, \|\nabla_u b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d,\mathbb{R}^{2d})}\}$. Combining these inequalities gives for (4.10):

$$\begin{aligned}
&\left|\check{M}_t^V(x,u) - \check{M}_t^V(\bar{x},u)\right| \le C_{\nabla_x b,\nabla_u b,T}\left(\int_0^t \left|\check{M}_s^V(x,u) - \check{M}_s^V(\bar{x},u)\right| ds + |x-\bar{x}|\right) \\
&\quad + C_{\nabla_x b,\nabla_u b,T}\int_0^t \left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)\left|\mathrm{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right) - \mathrm{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right| ds\,.
\end{aligned} \tag{4.19}$$

Taking the expectation under $\mathbb{P}_{z,\eta}$ of the previous equation, we obtain for any $p \geq 1$:

$$\mathbb{E}_{z,\eta}\left|\check{M}_t^V(x,u)-\check{M}_t^V(\bar{x},u)\right|^p \leq C^p_{\nabla_x b,\nabla_u b,T}3^{p-1}\left(\mathbb{E}_{z,\eta}\int_0^t\left|\check{M}_s^V(x,u)-\check{M}_s^V(\bar{x},u)\right|^p ds+|x-\bar{x}|^p\right.$$
$$\left.+\mathbb{E}_{z,\eta}\int_0^t\left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)^p\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)-\operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^p ds\right). \tag{4.20}$$

Gronwall's lemma gives that:

$$\mathbb{E}_{z,\eta}\left|\check{M}_t^V(x,u)-\check{M}_t^V(\bar{x},u)\right|^p \leq C_{\nabla_x b,\nabla_u b,T,p}e^{C_{\nabla_x b,\nabla_u b,T,p}}\left(|x-\bar{x}|\right.$$
$$\left.+\int_0^t\mathbb{E}_{z,\eta}\left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)^p\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)-\operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^p ds\right) \tag{4.21}$$

where $C_{\nabla_x b,\nabla_u b,T,p}$ depends on $\nabla_x b$, $\nabla_u b$, T and p. Recalling that components of the d dimensional drift-less free Langevin model in (4.8) are independent:

$$\begin{aligned}
&\mathbb{E}_{z,\eta}\left(1+\left|(\widetilde{\eta}_s^u)^{(d)}\right|\right)^p\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)-\operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^p\\
&\leq\left(\mathbb{E}_{z,\eta}\left(1+\left|u^{(d)}+\sigma\check{W}_t^{(d)}\right|\right)^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}_{z,\eta}\left|\operatorname{sign}\left((\widetilde{z}_s^{x,u})^{(d)}\right)-\operatorname{sign}\left((\widetilde{z}_s^{\bar{x},u})^{(d)}\right)\right|^{2p}\right)^{\frac{1}{2}}\\
&\leq 3^{p-\frac{1}{2}}\left(1+|u^{(d)}|^{2p}+t^p\sigma^{2p}\frac{2p!}{2^p p!}\right)^{\frac{1}{2}}\\
&\quad\times\left(\mathbb{E}_{z,\eta}\left|\operatorname{sign}\left(x^{(d)}+u^{(d)}s+\sigma\int_0^s\check{W}_\theta^{(d)}\,d\theta\right)-\operatorname{sign}\left(u^{(d)}s+\sigma\int_0^s\check{W}_\theta^{(d)}\,d\theta\right)\right|^{2p}\right)^{\frac{1}{2}}\\
&\leq C_{u,\sigma,T,p}\left(2^{2p}\mathbb{P}_{z,\eta}\left(u^{(d)}s+\sigma\int_0^s\check{W}_\theta^{(d)}\,d\theta\leq 0\leq x^{(d)}+u^{(d)}s+\sigma\int_0^s\check{W}_\theta^{(d)}\,d\theta\right)\right)^{\frac{1}{2}}\\
&\leq C_{u,\sigma,T,p}\left(\operatorname{erf}\left(\sqrt{\frac{3}{2}}\frac{x^{(d)}+su^{(d)}}{\sigma s\sqrt{s}}\right)-\operatorname{erf}\left(\sqrt{\frac{3}{2}}\frac{u}{\sigma\sqrt{s}}\right)\right)^{\frac{1}{2}}
\end{aligned} \tag{4.22}$$

where $C_{u,\sigma,T,p}$ depends on u, σ, T and p, and erf is the error function.

Lebesgue dominated convergence theorem gives that bound in (4.21) converges to 0 as x goes to $\bar{x}$. This shows:

$$\mathbb{E}_{z,\eta}\left|\check{M}_t^V(x,u)-\check{M}_t^V(\bar{x},u)\right|^p\longrightarrow 0,\quad\text{as}\quad x\to\bar{x}\in\partial\mathcal{D}.$$

For any $p\geq 1$:

$$\begin{aligned}
\mathbb{E}_{z,\eta}\left|\check{M}_t^Y(x,u)-\check{M}_t^Y(\bar{x},u)\right|^p&=\mathbb{E}_{z,\eta}\left|\int_0^t(\check{M}_s^V(x,u)-\check{M}_s^V(\bar{x},u))\,ds\right|^p\\
&\leq t^{p-1}\int_0^t\mathbb{E}_{z,\eta}\left|\check{M}_s^V(x,u)-\check{M}_s^V(\bar{x},u)\right|^p ds
\end{aligned}$$

and using Lebesgue convergence theorem and the previous convergence result, as $x\to\bar{x}$:

$$\mathbb{E}_{z,\eta}\left|\check{M}_t^Y(x,u)-\check{M}_t^Y(\bar{x},u)\right|^p\to 0.$$

Similar computations allow to show that for $x\to\bar{x}$:

$$\begin{aligned}
&\mathbb{E}_{z,\eta}\left|\check{N}_t^Y(x,u)-\check{N}_t^Y(\bar{x},u)\right|^p\to 0\\
&\mathbb{E}_{z,\eta}\left|\check{N}_t^V(x,u)-\check{N}_t^V(\bar{x},u)\right|^p\to 0.
\end{aligned}$$

■

Remark 4.4. *Following similar arguments as the ones presented in the proof of Lemma 4.3, we can show that for any* $(t,x) \in [0,T] \times \overline{\mathcal{D}}$ *and* $p \in [1,\infty)$*, the processes* $\check{M}_t^Y(x,\cdot)$*,* $\check{M}_t^V(x,\cdot)$*,* $\check{N}_t^Y(x,\cdot)$ *and* $\check{N}_t^V(x,\cdot)$ *are continuous on* $\mathbb{R}^D$ *in norm* L^p*.*

4.3 Application to the confined process

Girsanov transform

Consider the probability measure $\mathbb{P}_{x,u}$ defined by:

$$\left.\frac{d\mathbb{P}_{x,u}}{d\mathbb{P}_{z,\eta}}\right|_{\mathcal{F}_T} = G_T(x,u) := \exp\left(\int_0^T \widetilde{b}(z_s^{x,u},\eta_s^u)\, d\check{W}_s - \frac{1}{2}\int_0^T \widetilde{b}^2(z_s^{x,u},\eta_s^u)\, ds\right). \tag{4.23}$$

Since $\widetilde{b}$ is bounded, then, for any (x,u) in $\mathcal{D} \times \mathbb{R}^d$, $(G(x,u)_t)_{0\le t\le T}$ is a martingale and we have that $\mathbb{P}_{z,\eta} \sim \mathbb{P}_{x,u}$. By Girsanov's theorem, then the process $(z_t^{x,u},\eta_t^u)_{0\le t\le T}$ solves the equation (4.1) under $\mathbb{P}_{x,u}$. This also means that (4.9) under $\mathbb{P}_{z,\eta}$ is equal in distribution to (4.6) under $\mathbb{P}_{x,u}$.

By (H_{PDE})-(i) and (H_{PDE})-(ii), we have that the function $\widetilde{b}$ is Lipschitz. Since the drift is sufficiently regular and σ is a constant, by [Friedman, 2012], the stochastic flow process $(x,u) \mapsto (Y_t^{x,u}, V_t^{x,u})$ is well defined and we can consider the function $f\colon \mathcal{D}\times\mathbb{R}^d \mapsto \mathbb{R}$ defined as $f(t,x,u) := \mathbb{E}_{x,u}\overline{\psi}(Y_t^{x,u},V_t^{x,u})$ where $\overline{\psi}$ is a continuous extension of the function ψ for negative values of $x^{(d)}$:

$$\overline{\psi} : (x,u) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \psi((x',|x^{(d)}|),(u',\mathrm{sign}(x^{(d)})u^{(d)})). \tag{4.24}$$

According to [Bossy and Jabir, 2011], the process defined as $(\mathfrak{X}_t,\mathfrak{U}_t) = ((Y_t',|Y_t^{(d)}|),(V_t',(\mathrm{sign}(Y_t^{(d)})_+)\times V_t^{(d)})_{t\ge 0}$ is a weak solution of (1.1), so for any $(t,x,u) \in [0,T]\times\mathcal{D}\times\mathbb{R}^d$:

$$\begin{aligned}
f(t,x,u) &= \mathbb{E}_{x,u}[\overline{\psi}(Y_t^{x,u},V_t^{x,u})] = \mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{x,u}>0\}}\overline{\psi}(Y_t^{x,u},V_t^{x,u})] + \mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{x,u}\le 0\}}\overline{\psi}(Y_t^{x,u},V_t^{x,u})] \\
&= \mathbb{E}_{x,u}[\mathbb{1}_{\{Y_t^{(d)x,u}>0\}}\psi(Y_t^{x,u},V_t^{x,u})] + \mathbb{E}_{x,u}\left[\mathbb{1}_{\{Y^{(d)}{}_t^{x,u}\le 0\}}\psi\left((Y_t'^{x,u},-Y_t^{(d)x,u}),(V_t'^{x,u},-V_t^{(d)x,u}\right)\right] \\
&= \mathbb{E}_{x,u}\left[\psi\left((Y_t'^{x,u},|Y_t^{(d)x,u}|),(V_t'^{x,u},\mathrm{sign}(Y_t^{(d)x,u})V_t^{(d)x,u}\right)\right] \\
&= \mathbb{E}_{x,u}\left[\psi(\mathfrak{X}_t,\mathfrak{U}_t)\right] = \mathbb{E}_{x,u}[\psi(X_t^{x,u},U_t^{x,u})].
\end{aligned} \tag{4.25}$$

We now state the lemma that contains a first part of the regularity results of Theorem 2.1:

Lemma 4.5. *The function* F *defined in* (1.7) *belongs in* $\mathcal{C}([0,T];L^\infty(\mathcal{D}\times\mathbb{R}^d);\mathbb{R}^d) \cap \mathcal{C}([0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d;\mathbb{R})$.

Proof. By similar arguments to (4.25), we can show that for any $(t,x,u) \in Q_T$, we have the equality

$$\mathbb{E}_{x,u}[\overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})] = \mathbb{E}_{x,u}[\psi(X_T^{t,x,u},U_T^{t,x,u})]$$

and they both equal $F(t,x,u)$ by definition (1.7). $(Y_T^{t,x,u},V_T^{t,x,u})$ is the solution at time T of the SDE (4.1) such that at time t, $(Y_t^{t,x,u},V_t^{t,x,u}) = (x,u)$. By the hypotheses ($H_{PDE}$)-($i$) and ($H_{PDE}$)-($ii$), we have that the function $\widetilde{b}$ is Lipschitz (see Remark 1.4), therefore the flow $(t,x,u) \mapsto (Y_T^{t,x,u},V_T^{t,x,u})$ is almost surely continuous. The function ψ is continuous and bounded with support on $\mathcal{D}\times\mathbb{R}^d$, then $\overline{\psi}$ is also continuous and bounded. Let $(t,x,u) \in [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d$ then for $(t_k,x_k,u_k)_{k\in\mathbb{N}}$ such that $(t_k,x_k,u_k) \to (t,x,u)$, when $k\to\infty$, we have that $\overline{\psi}(Y_T^{t_k,x_k,u_k},V_T^{t_k,x_k,u_k}) \to \overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})$ a.s. Since $\overline{\psi}$ is bounded, then by the Dominated Convergence Theorem, $\mathbb{E}\left[\overline{\psi}(Y_T^{t_k,x_k,u_k},V_T^{t_k,x_k,u_k})\right] \to \mathbb{E}\left[\overline{\psi}(Y_T^{t,x,u},V_T^{t,x,u})\right]$, or written differently $F(t_k,u_k,x_k) \to F(t,x,u)$, when $(t_k,x_k,u_k) \to (t,x,u)$. This implies that F is continuous at $(t,x,u) \in [0,T]\times\overline{\mathcal{D}}\times\mathbb{R}^d$.

∎

Lemma 4.6. *The function F defined in* (1.7) *is such that* $\nabla_x F, \nabla_u F \in \mathcal{C}\left([0,T]; L^\infty(\mathcal{D})\right) \cap \mathcal{C}([0,T] \times \overline{\mathcal{D}} \times \mathbb{R}^d) \cap L^2(Q_T, \mathbb{R}^d) \cap L^2(\Sigma_T, \mathbb{R}^d)$.

Proof. Since for any $(t,x,u) \in Q_T$, $f(t,x,u) = \mathbb{E}_{x,u}[\psi(X_t^{x,u}, U_t^{x,u})]$, $F(t,x,u) = \mathbb{E}[\psi(X_T^{t,x,u}, U_T^{t,x,u})]$ and the fact that the process $(X_t, U_t)_{0 \leq t \leq T}$ is time-homogeneous (as the drift b is not time dependent), then is clear that $f(t,x,u) = F(T-t,x,u)$. Therefore if the regularity results from the statement of the lemma are proven for f, they will also apply to F. We work with the former in this proof.

Provided sufficient regularity on $\overline{\psi}$, we have that:

$$\nabla_x f(t,x,u) = \mathbb{E}_{x,u}\left[\nabla_x \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) M_t^Y(x,u) + \nabla_u \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) M_t^V(x,u)\right] \quad \text{a.e.} \tag{4.26}$$

and

$$\nabla_u f(t,x,u) = \mathbb{E}_{x,u}\left[\nabla_x \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) N_t^Y(x,u) + \nabla_u \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) N_t^V(x,u)\right] \quad \text{a.e.} \tag{4.27}$$

By hypothesis (H_{PDE})-(i) and Gronwall's lemma, there exists a constant $C_{\nabla_x b, \nabla_u b, T}$ depending on $\nabla_x b$, $\nabla_u b$ and T such that

$$\sup_{(t,x,u) \in \overline{Q_T}} \left(\left\|M_t^Y(x,u)\right\| + \left\|M_t^V(x,u)\right\| + \left\|N_t^Y(x,u)\right\| + \left\|N_t^V(x,u)\right\|\right) < C_{\nabla_x b, \nabla_u b, T}. \tag{4.28}$$

This result together with the boundedness of $\nabla_x \overline{\psi}$ and the bound (4.28) give that $\nabla_x f, \nabla_u f \in L^\infty(Q_t, \mathbb{R}^d)$.

Continuity of the derivatives

Let $(t,x,u) \in [0,T] \times \mathcal{D} \times \mathbb{R}^d$ and $\bar{x}$ the projection of x on $\partial \mathcal{D}$:

$$\begin{aligned}
&|\nabla_x f(t,x,u) - \nabla_x f(t,\bar{x},u)| = \left|\mathbb{E}_{x,u}\left[\nabla_x \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) M_t^Y(x,u) + \nabla_u \overline{\psi}(Y_t^{x,u}, V_t^{x,u}) M_t^V(x,u)\right]\right. \\
&\quad \left. - \mathbb{E}_{\bar{x},u}\left[\nabla_x \overline{\psi}(Y_t^{\bar{x},u}, V_t^{\bar{x},u}) M_t^Y(\bar{x},u) + \nabla_u \overline{\psi}(Y_t^{\bar{x},u}, V_t^{\bar{x},u}) M_t^V(\bar{x},u)\right]\right| \\
&= \left|\mathbb{E}_{z,\eta}\left[G_t(x,u)\left(\nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^Y(x,u) + \nabla_u \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^V(x,u)\right)\right]\right. \\
&\quad \left. - \mathbb{E}_{z,\eta}\left[G_t(\bar{x},u)\left(\nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) \check{M}_t^Y(\bar{x},u) + \nabla_u \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) \check{M}_t^V(\bar{x},u)\right)\right]\right| \\
&\leq \mathbb{E}_{z,\eta}\left|G_t(x,u) \nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^Y(x,u) - G_t(\bar{x},u) \nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) \check{M}_t^Y(\bar{x},u)\right| \\
&\quad + \mathbb{E}_{z,\eta}\left|G_t(x,u) \nabla_u \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^V(x,u) - G_t(\bar{x},u) \nabla_u \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) \check{M}_t^V(\bar{x},u)\right|.
\end{aligned} \tag{4.29}$$

Considering the first term:

$$\begin{aligned}
&\mathbb{E}_{z,\eta}\left|G_t(x,u) \nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^Y(x,u) - G_t(\bar{x},u) \nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) \check{M}_t^Y(\bar{x},u)\right| \\
&\leq \mathbb{E}_{z,\eta}\left|G_t(x,u) - G_t(\bar{x},u)\right| \left|\nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) \check{M}_t^Y(x,u)\right| \\
&\quad + \mathbb{E}_{z,\eta} G_t(\bar{x},u) \left|\check{M}_t^Y(x,u)\right| \nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) - \nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u) || \\
&\quad + \mathbb{E}_{z,\eta} G_t(\bar{x},u) \left|\nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u)\right| \left|\check{M}_t^Y(x,u) - \check{M}_t^Y(\bar{x},u)\right| \\
&\leq \left\|\nabla_x \overline{\psi}\right\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d, \mathbb{R}^d)} \left(\sup_{t,x,u \in Q_T} |\check{M}_t^Y(x,u)|\right) \mathbb{E}_{z,\eta}\left|G_t(x,u) - G_t(\bar{x},u)\right| \\
&\quad + \left(\sup_{t,x,u \in Q_T} |\check{M}_t^Y(x,u)|\right) \left(\mathbb{E}_{z,\eta} G_t(\bar{x},u)^2\right)^{\frac{1}{2}} \left(\mathbb{E}_{z,\eta}\left|\nabla_x \overline{\psi}(z_t^{x,u}, \eta_t^u) - \nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u)\right|^2\right)^{\frac{1}{2}} \\
&\quad + \left\|\nabla_x \overline{\psi}\right\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d, \mathbb{R}^d)} \left(\mathbb{E}_{z,\eta} G_t(\bar{x},u)^2\right)^{\frac{1}{2}} \left(\mathbb{E}_{z,\eta}\left|\nabla_x \overline{\psi}(z_t^{\bar{x},u}, \eta_t^u)\right| \left|\check{M}_t^Y(x,u) - \check{M}_t^Y(\bar{x},u)\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since the function $\widetilde{b}$ is Lipschitz by the hypotheses (H_{PDE})-(i) and (H_{PDE})-(ii), (see Remark 1.4) and, by their definitions in (4.8), for any $(t,u) \in [0,T] \times \mathbb{R}^d$, the function $x \mapsto (z_t^{x,u}, \eta_t^u)$ is continuous.

Then a.s. the function $x \mapsto G_t(x,u)$ for any $u \in \mathbb{R}^d$ is also continuous. For all $x \in \mathcal{D}$, $\mathbb{E}_{z,\eta} G_t(x,u) = \mathbb{E}_{z,\eta} G_t(\bar{x},u) = 1$, then the first term of the inequality goes to 0 as $x \to \bar{x}$ by Lebesgue Dominated Convergence theorem. Similarly, the Lipschitz continuity of $\partial_x \overline{\psi}$ and the L^p continuity of $(z_t^{x,u})_{0\le t\le T}$ in its initial condition implies that the second term converges to 0 as $x \to \bar{x}$. Also lemma 4.3 shows that the third term of the sum also goes to 0.

Similar arguments show that the second term of the bound in (4.29) goes to 0 as $x \to \bar{x}$, thus proving that $\nabla_x f(t,\cdot,u)$ is continuous up to the border $\partial\mathcal{D}$. And by repeating the same arguments, only replacing $\check{M}_t^Y(\cdot,u)$ and $\check{M}_t^V(\cdot,u)$ with $\check{N}_t^Y(\cdot,u)$ and $\check{N}_t^V(\cdot,u)$, we obtain also that $\nabla_u f(t,\cdot,u)$ is continuous up to the border $\partial\mathcal{D}$.

Through an analogous procedure that involves the continuity of $G_t(x,\cdot)$, the L^p continuity as expressed in the Remark 4.4 and boundedness on $\overline{Q_T}$ shown in (4.28) of $\check{M}_t^Y(x,\cdot)$, $\check{M}_t^V(x,\cdot)$, $\check{N}_t^Y(x,\cdot)$ and $\check{N}_t^V(x,\cdot)$, it can be shown that the functions $\nabla_x f(t,x,\cdot)$ and $\nabla_u f(t,x,\cdot)$ are continuous for any $(t,x) \in [0,T] \times \overline{\mathcal{D}}$ and the same for $\nabla_x f(\cdot,x,u)$ and $\nabla_u f(\cdot,x,u)$ for any $(x,u) \in \overline{\mathcal{D}} \times \mathbb{R}^d$.

Existence of the L^2 norms

Let $(t,x,u) \in \overline{Q_t}$, then

$$\begin{aligned}
|\nabla_x f(t,x,u)| &\le C_{\check{M}^Y,\check{M}^V} \left(\mathbb{E}_{z,\eta} G_t(x,u)^2\right)^{\frac{1}{2}} \left(\mathbb{E}_{z,\eta}\left(\left|\partial_x \overline{\psi}(z_t^{x,u},\eta_t^u)\right| + \left|\partial_u \overline{\psi}(z_t^{x,u},\eta_t^u)\right|\right)^2\right)^{\frac{1}{2}} \\
&\le C_{\check{M}^Y,\check{M}^V} e^{\frac{7}{4}T\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}} \left(\mathbb{E}_{z,\eta}\left(\left|\partial_x \overline{\psi}(z_t^{x,u},\eta_t^u)\right| + \left|\partial_u \overline{\psi}(z_t^{x,u},\eta_t^u)\right|\right)^2\right)^{\frac{1}{2}}
\end{aligned}$$

where $C_{\check{M}^Y,\check{M}^V} = \max\left\{\left\|\check{M}^Y\right\|_{L^\infty(Q_T,\mathbb{R}^{2d})}, \left\|\check{M}^V\right\|_{L^\infty(Q_T,\mathbb{R}^{2d})}\right\}$. Since $\overline{\psi} \in \mathcal{C}_c^{1,1}(\mathbb{R}^d \times \mathbb{R}^d)$, then we also have that $\nabla_x \overline{\psi} \in \mathcal{C}_c^{0,1}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\nabla_u \overline{\psi} \in \mathcal{C}_c^{0,1}(\mathbb{R}^d \times \mathbb{R}^d)$. So there exists two non-negative function $\beta_1, \beta_2 \colon \mathbb{R}^d \mapsto \mathbb{R}$ such that $\beta_1(x) = 1$ for $x \in \mathrm{Proj}_x\left(\mathrm{Supp}(\overline{\psi})\right)$ and 0 everywhere else, and $\beta_2(u) = 1$ for $u \in \mathrm{Proj}_u\left(\mathrm{Supp}(\overline{\psi})\right)$ and 0 everywhere else (where Proj_x and Proj_u are the projections according to the first d and the last d dimensions of $\mathbb{R}^{d\times d}$) and a constant

$$C = \sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} \left(\left|\nabla_x \overline{\psi}(x,u)\right| + \left|\nabla_u \overline{\psi}(x,u)\right|\right)^2$$

such that

$$\left(\left|\nabla_x \overline{\psi}(x,u)\right| + \left|\nabla_u \overline{\psi}(x,u)\right|\right)^2 \le C\beta_1(x)\beta_2(u) \implies |\nabla_x f(t,x,u)| \le C_{\check{M},T,b} \left(\mathbb{E}_{z,\eta}\beta_1(z_t^{x,u})\beta_2(\eta_t^u)\right)^{\frac{1}{2}}$$

where $C_{\check{M},T,b} = CC_{\check{M}^Y,\check{M}^V} e^{\frac{7}{4}T\|b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)}}$, so we can rewrite:

$$|\nabla_x f(t,x,u)| \le C_{\check{M},T,b} \left(\int_{\mathbb{R}^{2d}} \beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\, dz d\eta\right)^{\frac{1}{2}}$$

the function $\check{p}(t;\cdot,\cdot)$ being the density of the random variable $\left(\sigma\int_0^t \check{W}_s\,ds, \sigma\check{W}_t\right)$. Finally

$$\begin{aligned}\|\nabla_x f\|^2_{L^2(\Sigma_T,\mathbb{R}^d)} &= \int_{\Sigma_T} |(u\cdot n_{\mathcal{D}}(x))|\,|\nabla_x f(t,x,u)|^2\,dt\otimes d\sigma_{\partial\mathcal{D}}(x)\otimes du \\ &\leq C^2_{\check{M},T,b}\int_0^T\int_\Sigma\int_{\mathbb{R}^{2d}} |u|\beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\,dzd\eta dt\otimes d\sigma_{\partial\mathcal{D}}(x)\otimes du \\ &\leq C^2_{\check{M},T,b}\int_0^T dt\int_{\mathbb{R}^{2d}}\check{p}(t;z,\eta)\,dzd\eta\int_{\mathbb{R}^d}|u|\beta_2(u+\eta)\,du\int_{\partial\mathcal{D}}\beta_1(x+ut+z)\,d\sigma_{\partial\mathcal{D}}(x) \\ &\leq C^2_{\check{M},T,b}\lambda(\partial\mathcal{D}\cap\mathrm{Supp}(\overline{\psi}))\int_0^T dt\int_{\mathbb{R}^d}\check{p}(t;z,\eta)\,dzd\eta\left(\int_{\mathbb{R}^d}(|u+\eta|+|\eta|)\beta_2(u+\eta)\,du\right) \\ &\leq C^2_{\check{M},T,b}\lambda(\partial\mathcal{D}\cap\mathrm{Supp}(\overline{\psi}))\int_0^T dt\int_{\mathbb{R}^d}\frac{1}{(\sigma\sqrt{t})^d}p^{(d)}_{\mathcal{N}(0,I_d)}\left(\frac{\eta}{\sigma\sqrt{t}}\right)\,d\eta\left(C_{\beta_2}+|\eta|\lambda(\mathrm{Supp}(\overline{\psi}))\right)\end{aligned} \tag{4.30}$$

where $C_{\beta_2} = \int_{\mathbb{R}^d}|u|\beta_2(u)\,du$ is a constant and $p^{(d)}_{\mathcal{N}(0,I_d)}$ is the density of the centred $d-$dimensional normal distribution that has for covariance matrix I_d which admits moments of any order so the double integral left in the final equality is finite. Similar computations show that $\|\nabla_u f\|_{L^2(\Sigma_T)}$ is finite.

Now, we consider the norm:

$$\begin{aligned}\|\nabla_x f\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} &= \int_{\mathcal{D}\times\mathbb{R}^d}|\nabla_x f(t,x,u)|^2\;dtdxdu \\ &\leq C^2_{\check{M},T,b}\int_{\mathcal{D}}\int_{\mathbb{R}^{2d}}\beta_1(x+ut+z)\beta_2(u+\eta)\check{p}(t;z,\eta)\,dzd\eta dxdu \\ &\leq C^2_{\check{M},T,b}\int_{\mathbb{R}^{2d}}\check{p}(t;z,\eta)\,dzd\eta\int_{\mathbb{R}^d}\beta_2(u+\eta)\,du\int_{\mathcal{D}}\beta_1(x+ut+z)\,dx \\ &\leq C^2_{\check{M},T,b}\lambda\left(\mathrm{Supp}(\overline{\psi})\right)^2.\end{aligned}$$

Corollary 6.3 gives the result that $\nabla_u f\in L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)$. ■

5 Regularity of the Kolmogorov problem with specular boundary conditions

The bounds of the weak error (3.46) obtained in section 3 also depend on the $L^2(Q_T;\mathbb{R}^{2d})$ norms of $\mathrm{Hess}_{x,u}(F)$ and $\mathrm{Hess}_{u,u}(F)$ where F is the solution in distribution of (1.9), or under a probabilistic interpretation (1.7). This section focuses on this L^2 regularity of these second order derivatives, which is the final result of Theorem (2.1). Instead of working on this function, we consider the following $f\colon Q_T\mapsto\mathbb{R}$, $f(t,x,u) = \mathbb{E}\psi\left(X^{x,u}_t, U^{x,u}_t\right)$ for any $(t,x,u)\in Q_T$. As mentioned in the previous section, $f(t,x,u) = F(T-t,x,u)$, so the $L^2(Q_T,\mathbb{R}^{2d})$ regularity of the second order derivatives proven for one function, apply to the other. Again, we consider the former which verifies the equation (6.11).

Remark 5.1. *The solution f of (6.11) is in distribution, meaning that for any $\varphi \in \mathcal{C}_b^\infty(\overline{Q}_t)$ we have:*

$$\begin{aligned}
&\int_{Q_t} f(s,x,u)\left(\partial_s\varphi - (u\cdot\nabla_x\varphi) - (\nabla_u\cdot(b(x,u)\varphi)) + \frac{\sigma^2}{2}\triangle_u\varphi\right)(s,x,u)\,ds\,dx\,du\\
&= \int_{\mathcal{D}\times\mathbb{R}^d}\left[\varphi(s,x,u)f(s,x,u)\right]_{s=0}^{s=t}dx\,du - \int_{\Sigma_t^-}(u\cdot n_{\mathcal{D}}(x))\gamma(f)(s,x,u)\varphi(s,x,u)\,d\lambda_{\Sigma_T}(s,x,u)\\
&\quad - \int_{\Sigma_t^+}(u\cdot n_{\mathcal{D}}(x))\gamma(f)(s,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))\varphi(s,x,u)\,d\lambda_{\Sigma_T}(s,x,u),
\end{aligned} \tag{5.1}$$

Let us further notice that the trace function $\gamma(f)$ in $L^2(\Sigma_T)$ is characterized by the Green formula related to the transport operator $\partial_t + (u\cdot\nabla_x)$ (we refer to Subsection 3.1 for more details).

We extend the mollifiers defined in the introduction of Section 3 for $d\geq 2$. Let $(\widetilde{\beta}_k)_{k\geq1}$, $(\rho_n)_{n\geq1}$ and $(g_m)_{m\geq1}$ be smooth sequences such that:

$$\mathrm{Supp}(\widetilde{\beta}_k)\subset\left(-\frac{T}{k},0\right),\qquad \mathrm{Supp}(\rho_n)\subset\mathcal{B}_{\frac{1}{n}}(0;\mathbb{R}^{d-1})\times\left(-\frac{1}{n},0\right)\qquad\text{and}\qquad \mathrm{Supp}(g_m)=\mathbb{R}^d \tag{5.2}$$

where $\mathcal{B}_r\left(a;\mathbb{R}^{d-1}\right)$ is the $\mathbb{R}^{d-1}$ open ball centered at $a\in\mathbb{R}^{d-1}$ with radius r

The sequence $(\widetilde{\beta}_k)_{k\geq1}$ is defined using the mollifying sequence $(\beta_k)_{k\geq1}$ from Section 3. For any $t\in\mathbb{R}$, we state that $\widetilde{\beta}_k(t)=\beta_k(-t)$. So $\widetilde{\beta}_k$ is reflection according to the abscissa of β_k.

Recalling the notation $x=(x',x^{(d)})$, for any $x\in\mathbb{R}^d$, we consider the generating function:

$$\rho\colon x\mapsto\begin{cases}\exp\left(-\dfrac{1}{1-\|x'\|^2}\right)\exp\left(-\dfrac{1}{x^{(d)}(-1-x^{(d)})}\right) & \text{for } x\in\mathcal{B}_1(0;\mathbb{R}^{d-1})\times(-1,0)\ ,\\ 0\quad\text{otherwise}\,, & \end{cases}$$

then for any $n\geq1$, $\rho_n(x)=Cn^d\rho(nx)$ where C is such that $\displaystyle\int_{\mathbb{R}^d}\rho(x)\,dx=\frac{1}{C}$.

For the sequence $(g_m)_{m\geq1}$ we choose to use the Gaussian kernel:

$$g\colon u\in\mathbb{R}\mapsto\frac{1}{\sqrt{(2\pi)^d}}\exp\left(-\frac{\|u\|^2}{2}\right)$$

by taking $g_m(u)=m^dg(mu)$, with the property that

$$ug_m(u)=-\frac{1}{m^2}\nabla_ug_m(u). \tag{5.3}$$

We define the regularisation of f the solution in distribution of (6.11) as $f_{k,n,m}\colon(\tau,y,v)\in\overline{Q_T}\mapsto\mathbb{R}$ as

$$f_{k,n,m}(\tau,y,v)=\int_{Q_T}f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu. \tag{5.4}$$

Also defined is $f_{n,m}$, the regularisation of f only w.r.t. the spatial coordinates, defined for every $(s,y,v)\in\overline{Q_T}$ as:

$$f_{n,m}(s,y,v)=\int_{\mathcal{D}\times\mathbb{R}^d}f(s,x,u)\rho_n(y-x)g_m(v-u)\,dxdu\,. \tag{5.5}$$

In the following Lemma, we obtain the equality verified by $f_{k,n,m}$.

Lemma 5.2. *The function $f_{\delta,n,m}$ on the interior of Q_T satisfies the equality*

$$\begin{aligned}&-\partial_\tau f_{k,n,m}(\tau,y,v)+(v\cdot\nabla_y f_{k,n,m})(\tau,y,v)+\left(b(y,v)\cdot\nabla_v f_{k,n,m}\right)(\tau,y,v)+\frac{\sigma^2}{2}\triangle_v f_{k,n,m}(\tau,y,v)\\&=R_{k,n,m}[f](\tau,y,v).\end{aligned}\tag{5.6}$$

with

$$R_{k,n,m}[f](\tau,y,v)=R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)+R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)$$

where

$$\begin{aligned}R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)&\coloneqq f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\right)\\&\quad+\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\\R^{\mathrm{Tm}}_{k,n,m}[f](\tau,y,v)&\coloneqq\widetilde{\beta}_k(\tau-T)f_{n,m}(T,y,v)\,.\end{aligned}\tag{5.7}$$

Proof. To prove this Lemma, we consider a specific test function that is applied to the equation in Remark 5.1 and which gives the desired result.

We consider a test function $\varphi\in\mathcal{C}^\infty_b(\overline{Q}_T)$. The function $\hat{\varphi}_{k,n,m}\colon(s,x,u)\in Q_T\mapsto\hat{\varphi}_{k,n,m}(s,x,u)\in\mathbb{R}$ defined as:

$$\hat{\varphi}_{k,n,m}(s,x,u)=\int_{Q_T}\varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv$$

is in $\mathcal{C}^\infty_b(\overline{Q}_T)$ and $\hat{\varphi}_{k,n,m}$ vanished close to $\partial\mathcal{D}$ since the support of $\rho_n(y-\cdot)$ is in $\mathcal{D}$ for any $y\in\mathcal{D}$. We mention that the mollifying sequence $(\rho_n)_{n\geq1}$ has been chosen such that it removes the contribution of the boundary Σ_T in the equation (5.1). The Remark 5.1 applies for the test function $\hat{\varphi}_{k,n,m}(t,x,u)$ and on the whole domain Q_T we obtain:

$$\begin{aligned}&\int_{Q_T}f(s,x,u)\left(\partial_s\hat{\varphi}_{k,n,m}-(u\cdot\nabla_x\hat{\varphi}_{k,n,m})-\nabla_u\cdot(b(x,u)\hat{\varphi}_{k,n,m})+\frac{\sigma^2}{2}\triangle_u\hat{\varphi}_{k,n,m}\right)(s,x,u)\,dsdxdu\\&=\int_{\mathcal{D}\times\mathbb{R}^d}\left[\hat{\varphi}_{k,n,m}(s,x,u)f(s,x,u)\right]_{s=0}^{s=T}dxdu.\end{aligned}\tag{5.8}$$

By using Fubini's theorem, we pass the mollifiers on the function f in order to obtain the equality for function $f_{k,n,m}$.

We start by analysing every term of equation (5.8), one by one. The first term corresponds to the derivative in time, and by noticing that $\partial_s\widetilde{\beta}_k(\tau-s)=-\partial_\tau\widetilde{\beta}_k(\tau-s)$:

$$\begin{aligned}&\int_{Q_T}f(s,x,u)\partial_s\hat{\varphi}_{k,n,m}(s,x,u)\,dsdxdu\\&=\int_{Q_T}f(s,x,u)\partial_s\left(\int_{Q_T}\varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\right)dsdxdu\\&=\int_{Q_T\times Q_T}f(s,x,u)\varphi(\tau,y,v)\partial_s\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\,dsdxdu\\&=-\int_{Q_T\times Q_T}f(s,x,u)\varphi(\tau,y,v)\partial_\tau\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,d\tau dydv\,dsdxdu\\&=-\int_{Q_T}\varphi(\tau,y,v)\partial_\tau\left(\int_{Q_T}f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right)d\tau dydv\\&=-\int_{Q_T}\varphi(\tau,y,v)\partial_\tau f_{k,n,m}(\tau,y,v)\,d\tau dydv\end{aligned}$$

while the r.h.s term of equation (5.8) becomes:

$$
\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} \left[\hat{\varphi}_{k,n,m}(s,x,u)f(s,x,u)\right]_{s=0}^{s=T} dxdu \\
&= \int_{\mathcal{D}\times\mathbb{R}^d} \hat{\varphi}_{k,n,m}(T,x,u)f(T,x,u)\,dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} \hat{\varphi}_{k,n,m}(0,x,u)f(0,x,u)\,dxdu \\
&= \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau-T)\int_{\mathcal{D}\times\mathbb{R}^d} f(T,x,u)\rho_n(y-x)g_m(v-u)\,dxdud\tau dydv \\
&- \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau)\int_{\mathcal{D}\times\mathbb{R}^d} f(0,x,u)\rho_n(y-x)g_m(v-u)\,dxdud\tau dydv \\
&= \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau-T)f_{n,m}(T,y,v)\,d\tau dydv - \int_{Q_T} \varphi(\tau,y,v)\widetilde{\beta}_k(\tau)f_{n,m}(0,y,v)\,d\tau dydv \\
&= \int_{Q_T} \varphi(\tau,y,v)\left[\widetilde{\beta}_k(\tau-s)f_{n,m}(s,y,v)\right]_{s=0}^{s=T} d\tau dydv
\end{aligned} \tag{5.9}
$$

The term $s=0$ is zero since the support of $\widetilde{\beta}_k$ is included just on $[-T,0]$.

By Fubini's theorem, the term corresponding to the drift in x in equation (5.8) can be rewritten as:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(u\cdot\nabla_x\hat{\varphi}_{k,n,m}(s,x,u))\,dsdxdu \\
&= \int_{[0,T]^2\times\mathbb{R}^{2d}} \widetilde{\beta}_k(\tau-s)g_m(v-u)\,dsd\tau dudv \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_x\rho(y-x))\,dxdy
\end{aligned} \tag{5.10}
$$

and, for the sake of simplicity we develop just the inner integral. Since $\nabla_x\rho(y-x) = -\nabla_y\rho(y-x)$, we have that

$$
\begin{aligned}
&\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_x\rho_n(y-x))\,dxdy = -\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(u\cdot\nabla_y\rho_n(y-x))\,dxdy \\
&= -\int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)(v\cdot\nabla_y\rho_n(y-x))\,dxdy + \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)((v-u)\cdot\nabla_y\rho_n(y-x))\,dxdy \\
&= -\int_{\mathcal{D}} \varphi(\tau,y,v)\left(v\cdot\nabla_y\int_{\mathcal{D}} f(s,x,u)\rho_n(y-x)\,dx\right)dy + \int_{\mathcal{D}\times\mathcal{D}} f(s,x,u)\varphi(\tau,y,v)((v-u)\cdot\nabla_y\rho_n(y-x))\,dxdy
\end{aligned} \tag{5.11}
$$

This means that we can rewrite the term (5.10) as:

$$
\begin{aligned}
&\int_{Q_T} f(s,x,u)(u\cdot\nabla_x\hat{\varphi}_{k,n,m}(s,x,u))\,dsdxdu = -\int_{Q_T} \varphi(\tau,y,v)\left(v\cdot\nabla_y f_{k,n,m}\right)\,d\tau dydv \\
&\quad + \int_{Q_T} \varphi(\tau,y,v)\left(\int_{Q_T} f(s,x,u)((v-u)\cdot\nabla_y\rho_n(y-x))\widetilde{\beta}_k(\tau-s)g_m(v-u)\,dsdxdu\right)d\tau dydv
\end{aligned} \tag{5.12}
$$

For the term corresponding to the drift in u in equation (5.8), we can perform an i.b.p. because $f\in H^1(\mathbb{R}^d)$ in the variable u according to Lemma (4.6), we then add and subtract a term in $b(y,v)$, and perform and integration by parts on one of these terms to obtain:

$$\begin{aligned}
&\int_{Q_T} f(s,x,u)(\nabla_u\cdot(b(x,u)\hat{\varphi}_{k,n,m}(s,x,u)))\,dsdxdu = \int_{Q_T}\hat{\varphi}_{k,n,m}(s,x,u)(b(x,u)\cdot\nabla_u f(s,x,u))\,dsdxdu\\
&= \int_{Q_T^2} f(s,x,u)\varphi(\tau,y,v)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)(b(y,v)\cdot\nabla_u g_m(v-u))\,dsd\tau dxdydudv\\
&\quad -\int_{Q_T^2}(\nabla_u f(s,x,u)\cdot((b(x,u)-b(y,v)))\varphi(\tau,y,v)\rho_n(y-x)g_m(v-u)\widetilde{\beta}_k(\tau-s)\,dsd\tau dxdydudv.
\end{aligned} \tag{5.13}$$

As $\nabla_u g_m(v-u) = -\nabla_v g_m(v-u)$, we have, after several applications of Fubini's theorem:

$$\begin{aligned}
&\int_{Q_T} f(s,x,u)(\nabla_u\cdot(b(x,u)\hat{\varphi}_{k,n,m}))\,dsdxdu\\
&= -\int_{Q_T}\varphi(\tau,y,v)\left(b(y,v)\cdot\nabla_v f_{k,n,m}\right)\,dsdydv\\
&\quad -\int_{Q_T}\varphi(\tau,y,v)\left(\int_{Q_T}((b(x,u)-b(y,v))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right)d\tau dydv
\end{aligned} \tag{5.14}$$

The diffusion term becomes

$$\frac{\sigma^2}{2}\int_{Q_T} f(s,x,u)\triangle_u\hat{\varphi}_{k,n,m}(s,x,u)\,dsdxdu = \frac{\sigma^2}{2}\int_{Q_T}\varphi(\tau,y,v)\triangle_v f_{k,n,m}(\tau,y,v)\,d\tau dydv \tag{5.15}$$

The smoothed version $f_{k,n,m}$ on Q_T of the weak solution f of (5.1) verifies for any $\varphi\in C_b^\infty(\overline{Q}_T)$:

$$\begin{aligned}
&-\int_{Q_T}\varphi(\tau,y,v)\partial_\tau f_{k,n,m}(\tau,y,v)\,d\tau dydv + \int_{Q_T}\varphi(\tau,y,v)\left(v\cdot\nabla_y f_{k,n,m}\right)\,d\tau dydv\\
&\quad +\int_{Q_T}\varphi(\tau,y,v)\left(b(y,v)\cdot\nabla_v f_{k,n,m}\right)\,dsdydv + \frac{\sigma^2}{2}\int_{Q_T}\varphi(\tau,y,v)\triangle_v f_{k,n,m}(\tau,y,v)\,d\tau dydv\\
&= \int_{Q_T}\varphi(\tau,y,v)\left[\widetilde{\beta}_k(\tau-s)f_{n,m}(s,y,v)\right]_{s=0}^{s=T}\,d\tau dydv\\
&\quad +\int_{Q_T}\varphi(\tau,y,v)\left(\int_{Q_T} f(s,x,u)((v-u)\cdot\nabla_y\rho_n(y-x))\widetilde{\beta}_k(\tau-s)g_m(v-u)\,dsdxdu\right)\,d\tau dydv\\
&\quad +\int_{Q_T}\varphi(\tau,y,v)\left(\int_{Q_T}((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right)d\tau dydv
\end{aligned} \tag{5.16}$$

where $f_{n,m}$ is defined in (5.5).

We have that $\mathrm{Supp}(\widetilde{\beta}_k)\subset[-T,0]$ so for any $\tau\in[0,T]$, $\widetilde{\beta}_k(t)=0$ and since $f_{k,n,m}$ is a smooth function in the interior of Q_T, we obtain that

$$\begin{aligned}
&-\partial_\tau f_{k,n,m}(\tau,y,v) + (v\cdot\nabla_y f_{k,n,m})(\tau,y,v) + \left(b(y,v)\cdot\nabla_v f_{k,n,m}\right)(\tau,y,v) + \frac{\sigma^2}{2}\triangle_v f_{k,n,m}(\tau,y,v)\\
&= R_{k,n,m}[f](\tau,y,v).
\end{aligned} \tag{5.17}$$

■

Lemma 5.3. *Consider a function f such that $f, \nabla_u f, \nabla_x f \in \mathcal{C}([0,T]; L^\infty(\mathcal{D})) \cap \mathcal{C}([0,T] \times \overline{\mathcal{D}} \times \mathbb{R}^d) \cap L^2(Q_T; \mathbb{R}^d) \cap L^2(\Sigma_T, \mathbb{R}^d)$ and define for any $(\tau, y, v) \in Q_T$*

$$
\begin{aligned}
&R^{\mathrm{Sp}}_{k,n,m}[f](\tau, y, v) \\
&:= f * \left(\widetilde{\beta}_k(\nabla_y \rho_n \cdot (v g_m))\right) + \int_{Q_T} \left((b(y,v) - b(x,u)) \cdot \nabla_u f(s,x,u)\right) \widetilde{\beta}_k(\tau - s) \rho_n(y-x) g_m(v-u)\, ds dx du
\end{aligned}
\tag{5.18}
$$

and

$$
R^{\mathrm{Tm}}_{k,n,m}[f](\tau, y, v) := \widetilde{\beta}_k(\tau - T) f_{n,m}(T, y, v) \tag{5.19}
$$

By considering $n \sim m$ at infinity, then:

i) $\left\| R^{\mathrm{Sp}}_{k,n,m}[f](\tau, y, v) \right\|_{L^\infty(Q_T)} \xrightarrow[(k,n,m)\to\infty]{} 0$

ii) $\left\| \nabla_y R^{\mathrm{Sp}}_{k,n,m}[f](\tau, y, v) \right\|_{L^\infty(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f, b}$ *and* $\left\| \nabla_y R^{\mathrm{Sp}}_{k,n,m}[f] \right\|_{L^2(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f, b}$*, uniformly in (k,n,m)*

iii) $\left\| \nabla_v R^{\mathrm{Sp}}_{k,n,m}[f](\tau, y, v) \right\|_{L^\infty(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f, b}$ *and* $\left\| \nabla_v R^{\mathrm{Sp}}_{k,n,m}[f] \right\|_{L^2(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f, b}$ *uniformly in (k,n,m)*

iv) $\left| f(T,y,v) - \int_0^T R^{\mathrm{Tm}}_{k,n,m}[f](\tau, y, v)\, d\tau \right| \xrightarrow[(n,m)\to\infty]{} 0$

v) $\int_0^T \left\| \nabla_u R^{\mathrm{Tm}}_{k,n,m}[f](\tau, y, v) \right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq C$ *and* $\int_0^T \left\| \nabla_x R^{\mathrm{Tm}}_{k,n,m}[f](\tau, y, v) \right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq C$ *uniformly in (k,l,m).*

Proof.

For the proof of this Lemma we utilise several properties on the mollifiers $(\rho_n)_{n\geq 1}$ and $(g_m)_{m\geq 1}$ defined at the beginning of this section. We have that

$$
\int_{\mathbb{R}^d} \|x\| \rho_n(x)\, dx = \int_{\mathrm{Supp}(\rho_n)} \|x\| \rho_n(x)\, dx \leq \frac{1}{n} \int_{\mathrm{Supp}(\rho_n)} \rho_n(x)\, dx = \frac{1}{n}
$$

then

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\mathrm{Hess}_{x,x}(\rho_n)(x)|\, dx = C n^d \int_{\mathbb{R}^d} |\mathrm{Hess}_{x,x}(\rho(nx))|\, dx \\
&= C n^2 n^d \int_{\mathbb{R}^d} |\mathrm{Hess}_{x,x}(\rho)(nx)|\, dx = C n^2 \int_{\mathbb{R}^d} |\mathrm{Hess}_{x,x}(\rho)(x)|\, dx \leq C_{\nabla_x^2 \rho_1} n^2
\end{aligned}
$$

where $C_{\nabla_x^2 \rho_1}$ depends on C the integral of ρ and on the integral of the Hessian of ρ. Finally we have that

$$
\int_{\mathbb{R}^d} \|x\| \, |\nabla_x \rho_n(x)|\, dx \leq \frac{1}{n} n^d n \int_{\mathbb{R}^d} |\nabla_x \rho(nx)|\, dx \leq \int_{\mathbb{R}^d} |\nabla_x \rho(x)|\, dx
$$

Similar properties are deduced for $(g_m)_{m\geq 1}$.

***i)* Convergence of the error.**

We consider the first term of $R^{\mathrm{Sp}}_{k,n,m}[f]$ and the property (5.3):

$$\begin{aligned}\left| f * \left(\widetilde{\beta}_k(\nabla_y \rho_n \cdot (v g_m))\right)(\tau,y,v)\right| &= \frac{1}{m^2}\left| f * \left(\widetilde{\beta}_k(\nabla_y \rho_n \cdot (\nabla_v g_m))\right)(\tau,y,v)\right| \\ &= \frac{1}{m^2}\left| \nabla_v f * \left(\widetilde{\beta}_k \nabla_y \rho_n g_m\right)(\tau,y,v)\right| \\ &\leq \frac{n}{m^2}\|\nabla_v f\|_{L^\infty(Q_T;\mathbb{R}^d)} C_{\nabla_x \rho_1}\end{aligned} \tag{5.20}$$

where $C_{\nabla_x \rho_1}$ depends only on the gradient of ρ_1.

Since the function b is Lipschitz by hypothesis ($H_{Langevin}$)-(ii) and $\mathrm{Supp}(\rho_n) \subset \mathcal{B}_{\frac{1}{n}}(0;\mathbb{R}^{d-1}) \times (-\frac{1}{n}, 0)$

$$\begin{aligned}&\left|\int_{Q_T} ((b(y,v)-b(x,u))\cdot \nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right| \\ &\leq L_b \|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)} \int_{Q_T} (\|y-x\|+\|u-v\|)\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu \\ &\leq L_b \|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\int_{\mathcal{D}} \|y-x\|\,\rho_n(y-x)\,dx + \int_{\mathbb{R}^d}\|v-u\|\,g_m(v-u)\,du\right) \\ &\leq L_b \|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\frac{1}{n}+\frac{C_{g_1}}{m}\right)\end{aligned} \tag{5.21}$$

where C_{g_1} depends only $\int_{\mathbb{R}^d}\|u\|\,g_1(u)\,du$. Therefore we have that:

$$\left\|R^{\mathrm{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T)} \leq C_{\nabla_u f, b,\rho_1,g_1}\left(\frac{n}{m^2}+\frac{1}{m}+\frac{1}{n}\right) \tag{5.22}$$

***ii)* Bound on the derivative of the error in y.**

We consider the first term:

$$\begin{aligned}&\left\|\nabla_y\left(f * \left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(v g_m))\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} = \frac{1}{m^2}\left\|\nabla_y\left(f * \left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot \nabla_v g_m)\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \\ &= \frac{1}{m^2}\left\|\nabla_v f * \left(\widetilde{\beta}_k \mathrm{Hess}_{y,y}(\rho_n) g_m\right)(\tau,y,v)\right\| L^\infty(Q_T;\mathbb{R}^d) \leq C_{\nabla^2_y\rho_1}\|\nabla_u f(s,x,u)\|_{L^\infty(Q_T;\mathbb{R}^d)}\frac{n^2}{m^2}\end{aligned} \tag{5.23}$$

where $C_{\nabla^2_y\rho_1}$ depends only on the Hessian of ρ_1.

The second term is bounded using similar arguments as before concerning the fact that b is with bounded derivatives

$$\begin{aligned}&\left\|\nabla_y \int_{Q_T}((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \\ &\leq \left\|\nabla_y b(y,v)\cdot\int_{Q_T}\nabla_u f(s,x,u)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \\ &\quad + \left\|\int_{Q_T}((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u))\,\widetilde{\beta}_k(\tau-s)\nabla_y\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \\ &\leq \|\nabla_u f\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\|\nabla_y b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} + C_{\nabla_y\rho_1} + C_{g_1}\frac{n}{m}\right)\end{aligned} \tag{5.24}$$

Therefore we have that:

$$\left\|\partial_y R^{\text{Sp}}_{k,n,m}[f](\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} \leq C_{\nabla_u f,b,\rho_1,g_1}\left(1+\frac{n}{m}+\frac{n^2}{m^2}\right) \tag{5.25}$$

Through a similar procedure, taking the $L^2(Q_T;er^d)-$norm instead of the $L^\infty(Q_T;\mathbb{R}^d)-$norm on $\nabla_u f$, we obtain the desired result.

iii) **Bound on the derivative of the error in v.**

We consider the first term:

$$\begin{aligned}&\left\|\nabla_v\left(f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot(vg_m))\right)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)} = \frac{1}{m^2}\left\|\nabla_v f*\left(\widetilde{\beta}_k(\nabla_y\rho_n\cdot\nabla_v g_m)\right)(\tau,y,v)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\\ &\leq C_{\nabla_y\rho_1}C_{\nabla_v g_1}\left\|\nabla_u f(s,x,u)\right\|_{L^\infty(Q_T;\mathbb{R}^d)}\frac{n}{m}\end{aligned} \tag{5.26}$$

where $C_{\nabla_y\rho_1}$ depends only on the gradient of ρ_1 and $C_{\nabla_v g_1}$ on the gradient of g_1.

Following similar calculations in determining the previous bound for the derivative in y, we have that

$$\begin{aligned}&\left\|\nabla_v\int_{Q_T}\left((b(y,v)-b(x,u))\cdot\nabla_u f(s,x,u)\right)\widetilde{\beta}_k(\tau-s)\rho_n(y-x)g_m(v-u)\,dsdxdu\right\|\\ &\leq\|\nabla_u f\|_{L^\infty(Q_T;\mathbb{R}^d)}\left(\|\nabla_v b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}+C_{\rho_1,\nabla_v g_1}\frac{m}{n}+C_{\nabla_v g_1}\right)\end{aligned} \tag{5.27}$$

where $C_{\rho_1,\nabla_v g_1}$ depends on ρ_1 and on the derivative of g_1. Thus we conclude that

$$\left\|\partial_v R^{\text{Sp}}_{k,n,m}[f](\tau,y,v)\right\| \leq C_{\nabla_u f,b,\rho_1,g_1}\left(1+\frac{n}{m}+\frac{m}{n}\right) \tag{5.28}$$

Through a similar procedure, taking the $L^2(Q_T;\mathbb{R}^d)-$norm instead of the $L^\infty(Q_T;\mathbb{R}^d)-$norm on $\nabla_u f$, we obtain the desired second bound result.

iv) **Limit of the error.** We have for any $(\tau,y,v)\in Q_T$:

$$\begin{aligned}&\left|f(T,y,v)-\int_0^T R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\,d\tau\right| = \left|f(T,y,v)-\int_0^T\widetilde{\beta}(\tau-T)f_{n,m}(T,y,v)\,d\tau dydv\right|\\ &=\left|f(T,y,v)-f_{n,m}(T,y,v)\,d\tau dydv\right|\xrightarrow[(n,m)\to\infty]{}0\end{aligned} \tag{5.29}$$

v) **Bounds of the derivative of the error.**

$$\int_0^T\left\|\nabla_v R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} = \int_0^T\widetilde{\beta}(\tau-T)\left\|\nabla_v f_{n,m}(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq \left\|\nabla_v f(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \tag{5.30}$$

and similarly

$$\int_0^T\left\|\nabla_y R^{\text{Tm}}_{k,n,m}[f](\tau,y,v)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} = \int_0^T\widetilde{\beta}(\tau-T)\left\|\nabla_y f_{n,m}(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \leq \left\|\nabla_y f(T,\cdot,\cdot)\right\|_{L^2(\mathcal{D}\times\mathbb{R};\mathbb{R}^d)} \tag{5.31}$$

■

Lemma 5.4. *Assume (H_{PDE}). The weak solution f in $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ to equation (6.11) verifies*

i) $\mathrm{Hess}_{x,u}(f)\in L^2(Q_T;\mathbb{R}^{2d})$,

ii) $\mathrm{Hess}_{u,u}(f)\in L^2(Q_T;\mathbb{R}^{2d})$.

Proof. The proof for these results is based on the equality on $f_{k,n,m}$ from Lemma 5.2. By using an energy equality approach, we obtain a uniform bound in (k,n,m) for $\mathrm{Hess}_{x,u}(f_{k,n,m})$ and we utilise a result from Berzis to conclude.

i) Hessian in x,u

Since $f_{k,n,m}$ is a smooth function on Q_T, we differentiate equality (5.17) with respect to coordinate y_i where y_i is the $i-$th coordinate, to obtain:

$$\begin{aligned}&-\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)+(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)+\left(\partial_{y_i}b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\\&\quad+\left(b(y,v)\cdot\nabla_v\partial_{y_i}f_{k,n,m})\right)(\tau,y,v)+\frac{\sigma^2}{2}\triangle_v\partial_{y_i}f_{k,n,m}(\tau,y,v)=\partial_{y_i}R_{k,n,m}[f](\tau,y,v).\end{aligned}\tag{5.32}$$

We now multiply this equality by $\partial_{y_i}f_{k,n,m}$ and integrate on Q_T thus obtaining:

$$\begin{aligned}&-\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)\,d\tau dydv\\&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\left(\partial_{y_i}b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dydv\\&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot\nabla_v\partial_{y_i}f_{k,n,m})\right)(\tau,y,v)\,d\tau dydv\\&+\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\frac{\sigma^2}{2}\triangle_v\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv\\&=\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_{y_i}R_{k,n,m}[f](\tau,y,v)\,d\tau dydv\end{aligned}\tag{5.33}$$

We now consider each of term of the equation (5.33), starting with the time derivative term

$$\begin{aligned}&\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\partial_\tau\partial_{y_i}f_{k,n,m}(\tau,y,v)\,d\tau dydv=\frac{1}{2}\int_{Q_T}\partial_\tau\left(\partial_{y_i}f_{k,n,m}\right)^2(\tau,y,v)\,d\tau dydv\\&=\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i}f_{k,n,m}\right)^2(T,y,v)\,d\tau dydv-\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i}f_{k,n,m}\right)^2(0,y,v)\,d\tau dydv\,.\end{aligned}\tag{5.34}$$

The second term of equation (5.33) is such that

$$\begin{aligned}\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y\partial_{y_i}f_{k,n,m})(\tau,y,v)d\tau dydv&=\frac{1}{2}\int_{Q_T}\left(v\cdot\nabla_y\left(\partial_{y_i}f_{k,n,m}\right)^2\right)(\tau,y,v)d\tau dydv\\&=\frac{1}{2}\int_{\Sigma_T}\left(v\cdot n_{\mathcal{D}(x)}\right)\|\partial_{y_i}f_{k,n,m}\|^2\end{aligned}$$

The third term is left as is while the forth term of (5.33) is modified as

$$\begin{aligned}&\int_{Q_T}\partial_{y_i}f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot\nabla_v\partial_{y_i}f_{k,n,m})\right)(\tau,y,v)\,d\tau dydv\\&=\frac{1}{2}\int_{Q_T}\left(b(y,v)\cdot\nabla_v\left(\partial_{y_i}f_{k,n,m}\right)^2\right)(\tau,y,v)d\tau dydv=-\frac{1}{2}\int_{Q_T}\left(\nabla_v\cdot b\right)\left(\partial_{y_i}f_{k,n,m}\right)^2(\tau,y,v)d\tau dydv\end{aligned}$$

while for the Laplacian term in (5.33) we have that

$$\int_{Q_T} \partial_{y_i} f_{k,n,m}(\tau,y,v)(\nabla_v\cdot\nabla_v)\partial_{y_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv = -\int_{Q_T} \left\|\nabla_v \partial_{y_i} f_{k,n,m}\right\|^2(\tau,y,v)\,d\tau dy dv\,.$$

Finally, we rewrite (5.33) as

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i} f_{k,n,m}\right)^2(T,y,v)\,d\tau dy dv + \frac{1}{2}\int_{\mathcal{D}\times\mathbb{R}^d}\left(\partial_{y_i} f_{k,n,m}\right)^2(0,y,v)\,d\tau dy dv \\
&+\frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\partial_{y_i} f_{k,n,m}\right\|^2 + \int_{Q_T}\partial_{y_i} f_{k,n,m}(\tau,y,v)\left(\partial_{y_i} b(y,v)\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \frac{\sigma^2}{2}\int_{Q_T}\left\|\nabla_v\partial_{y_i} f_{k,n,m}\right\|^2(\tau,y,v)\,d\tau dy dv \\
&= \int_{Q_T}\partial_{y_i} f_{k,n,m}(\tau,y,v)\partial_{y_i} R_{k,n,m}[f](\tau,y,v)\,d\tau dy dv\,.
\end{aligned} \tag{5.35}$$

Summing this equation from $i=1$ to $i=d$ and recalling that

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\partial_{v_j}\partial_{y_i} f_{k,n,m}\right)^2 = \|\mathrm{Hess}_{y,v} f_{k,n,m}\|^2,$$

we obtain that:

$$\begin{aligned}
&-\frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\nabla_y f_{k,n,m}\right\|^2 \\
&+\int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \\
&= \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\,.
\end{aligned} \tag{5.36}$$

which we reorganise as

$$\begin{aligned}
\frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} &= -\frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) \\
&+\frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\nabla_y f_{k,n,m}\right\|^2 + \int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\,.
\end{aligned} \tag{5.37}$$

We have the following inequality:

$$\begin{aligned}
\frac{\sigma^2}{2}\left\|\mathrm{Hess}_{y,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} &\le \frac{1}{2}\left\|\nabla_y f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\nabla_y f_{k,n,m}\right\|^2 \\
&+\int_{Q_T}\left((\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_y f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\,.
\end{aligned} \tag{5.38}$$

and we bound each of the terms in the r.h.s. of (5.38), uniformly in (k, l, m) using the regularity of the function f obtained from Lemma 4.6

$$\|\nabla_y f_{k,n,m}\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) \leq \sup_{t\in[0,T]} \|\nabla_y f_{n,m}\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(t) \leq \sup_{t\in[0,T]} \|\nabla_y f\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(t)\,.$$

By (H_{PDE}), the derivatives of the function b are bounded, then by Cauchy-Schwartz:

$$\begin{aligned}
&\left|\int_{Q_T} \left(\left(\nabla_y f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_y(b)(y,v)\right)\cdot \nabla_v f_{k,n,m}\right)\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\leq \|\mathrm{Jac}_y(b)\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\|\nabla_y f_{k,n,m})\|_{L^2(Q_T;\mathbb{R}^d)}\|\nabla_v f_{k,n,m})\|_{L^2(Q_T;\mathbb{R}^d)} \\
&\leq \|\mathrm{Jac}_y(b)\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^{2d})}\|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)}\|\nabla_v f\|_{L^2(Q_T;\mathbb{R}^d)}
\end{aligned}$$

while

$$\left|\int_{Q_T} (\nabla_v\cdot b)\,\|\nabla_y f_{k,n,m}\|^2(\tau,y,v)d\tau dy dv\right| \leq \|\nabla_y\cdot b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}\|\nabla_y f\|^2_{L^2(Q_T;\mathbb{R}^d)}\,.$$

We now consider Lemma 5.3 to control the errors as:

$$\begin{aligned}
&\left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\leq \left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R^{\mathrm{Sp}}_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| + \left|\int_{Q_T} \left(\nabla_y f_{k,n,m}\cdot \nabla_y R^{\mathrm{Tm}}_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\right| \\
&\leq \|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)}\left\|\nabla_y R^{\mathrm{Sp}}_{k,n,m}[f]\right\|_{L^2(Q_T;\mathbb{R}^d)} + \sup_{t\in[0,T]}\|\nabla_y f_{k,n,m}(t,\cdot,\cdot)\|_{L^2(\mathcal{D}\times\mathbb{R}^2;\mathbb{R}^d)}\int_0^T \left\|\nabla_y R^{\mathrm{Tm}}_{k,n,m}[f]\right\|_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)} \\
&\leq C_{\nabla_y f}\|\nabla_y f\|_{L^2(Q_T;\mathbb{R}^d)} + C_{\nabla_y f}\sup_{t\in[0,T]}\|\nabla_y f\|_{L^2(\mathcal{D}\times\mathbb{R}^D;\mathbb{R}^d)}\,.
\end{aligned} \tag{5.39}$$

By combining these various bounds and going back to inequality (5.38) we obtain that

$$\frac{\sigma^2}{2}\|\mathrm{Hess}_{y,v}(f_{k,n,m})\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \leq C_{\nabla_y f,\nabla_v f,b,\nabla_y b} + \frac{1}{2}\|\nabla_y f_{k,n,m}\|^2_{L^2(\Sigma_T)} \tag{5.40}$$

where $C_{\nabla_y f,\nabla_v f,b,\nabla_y b}$ does not depend on (k, n, m). By Lemma 4.6, we have that $\|\nabla_y f\|^2_{L^2(\Sigma_T)}$ is finite, therefore $\mathrm{Hess}_{y,v}(f_{k,n,m})$ is bounded in $L^2(Q_T;\mathbb{R}^{2d})$. Since $\nabla_y f_{k,l,m}$ and $\nabla_v f_{k,l,m}$ converge in $L^2(Q_T;\mathbb{R}^d)$, by [Brezis, 2010], we obtain that $\mathrm{Hess}_{y,v}(f) \in L^2(Q_T;\mathbb{R}^{2d})$.

ii) Hessian in u, u

We now prove a similar result for the second derivative w.r.t. u. We apply the same calculations: differentiate equality (5.17) with respect to coordinate v_i where v_i is the $i-$th coordinate, multiplying by $\partial_{v_i} f_{k,n,m}$ and integrating over Q_T, we obtain:

$$
\begin{aligned}
&-\int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_\tau \partial_{v_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_{y_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv \\
&\quad + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)(v\cdot\nabla_y \partial_{v_i} f_{k,n,m})(\tau,y,v)\,d\tau dy dv \\
&\quad + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\left(\partial_{v_i} b(y,v)\cdot \nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&\quad + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\left(b(y,v)\cdot \nabla_v \partial_{v_i} f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&\quad + \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\frac{\sigma^2}{2}\triangle_v \partial_{v_i} f_{k,n,m}(\tau,y,v)\,d\tau dy dv \\
&= \int_{Q_T} \partial_{v_i} f_{k,n,m}(\tau,y,v)\partial_{v_i} R_{k,n,m}[f](\tau,y,v)\,d\tau dy dv\,.
\end{aligned}
\tag{5.41}
$$

Now we sum for $i=1$ to $i=d$ and integrate by parts as in the previous section to obtain that

$$
\begin{aligned}
&-\frac{1}{2}\left\|\nabla_v f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(T) + \frac{1}{2}\left\|\nabla_v f_{k,n,m}\right\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d)}(0) + \int_{Q_T}\left(\nabla_y f_{k,n,m}\cdot\nabla_v f_{k,n,m}\right)(\tau,y,v)\,d\tau dy dv \\
&+\frac{1}{2}\int_{\Sigma_T}(v\cdot n_{\mathcal{D}(x)})\left\|\nabla_v f_{k,n,m}\right\|^2 + \int_{Q_T}\left((\nabla_v f_{k,n,m}(\tau,y,v)\cdot \mathrm{Jac}_v(b)(y,v))\cdot\nabla_v f_{k,n,m})\right)(\tau,y,v)\,d\tau dy dv \\
&-\frac{1}{2}\int_{Q_T}(\nabla_v\cdot b)\left\|\nabla_v f_{k,n,m}\right\|^2(\tau,y,v)d\tau dy dv - \frac{\sigma^2}{2}\left\|\mathrm{Hess}_{v,v}(f_{k,n,m})\right\|^2_{L^2(Q_T;\mathbb{R}^{2d})} \\
&= \int_{Q_T}\left(\nabla_v f_{k,n,m}\cdot\nabla_v R_{k,n,m}[f]\right)(\tau,y,v)\,d\tau dy dv\,.
\end{aligned}
\tag{5.42}
$$

By using the analogous arguments as previously, we obtain that $\mathrm{Hess}_{v,v}(f_{k,n,m})$ is bounded in $L^2(Q_T;\mathbb{R}^{2d})$ and since $\nabla_v f_{k,n,m}$ converges in $L^2(Q_T;\mathbb{R}^d)$, we obtain by [Brezis, 2010], that $\mathrm{Hess}_{v,v}(f)\in L^2(Q_T;\mathbb{R}^{2d})$.

■

Corollary 5.5. *Assume (H$_{PDE}$). The weak solution F to equation (1.9) verifies that* $\mathrm{Hess}_{x,u}(F),\mathrm{Hess}_{u,u}(F)\in L^2(Q_T;\mathbb{R}^{2d})$.

Proof. By the previous lemma, we have that $\mathrm{Hess}_{x,u}(f),\mathrm{Hess}_{u,u}(f)\in L^2(Q_T,\mathbb{R}^{2d})$. And since for any $(t,x,u)\in Q_T$, $f(t,x,u)=F(T-t,x,u)$, we have that:

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(f)\right\|^2_F(t,x,u)\,dt dx du = \int_0^T\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(F)\right\|^2_F(T-t,x,u)\,dt dx du \\
&= -\int_T^0\int_{\mathcal{D}\times\mathbb{R}^d}\left\|\mathrm{Hess}_{x,u}(F)\right\|^2_F(s,x,u)\,ds dx du = \left\|\mathrm{Hess}_{x,u}(F)\right\|^2_{L^2(Q_T,\mathbb{R}^{2d})} < +\infty
\end{aligned}
$$

by performing the change of variable $s=T-t$. The same argument gives that $\mathrm{Hess}_{u,u}(F)\in L^2(Q_T;\mathbb{R}^{2d})$.

■

6 On the semigroup of the confined Langevin process

In this section we present several results that pertain to the existence and regularity of the weak solution of the PDE (1.9). Without any loss of generality, we consider the time forward formulation of this PDE, written is its variational formulation in (5.1). This section is an extract from [Bossy and Jabir, 2015] with minor modifications to include a bounded Lipschitz drift b in the PDE problem (6.5). We first assume that b is a smooth function and then we come back to our hypothesis (H_{PDE}). The proofs are transferred to the Appendix3.

We investigate some estimates related to the semigroup associated to the solution of the SDE (1.1); namely, for a test function $\psi \in \mathcal{C}_c^\infty(\mathcal{D}\times\mathbb{R}^d)$, for all $(x,u)\in(\mathcal{D}\times\mathbb{R}^d)\cup(\Sigma\setminus\Sigma^0)$, we define

$$\Gamma^\psi(t,x,u) := \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u},U_t^{x,u})\right], \tag{6.1}$$

where $((X_t^{x,u},U_t^{x,u});\, t\in[0,T])$ is the solution of (1.1) starting from $(0,x,u)$ and $((X_t^{s,x,u},U_t^{s,x,u});\, t\in[0,T])$ is the solution of (1.1) starting from (s,x,u).

Pathwise uniqueness of the confined Langevin process implies that for all $0\le s\le t\le T$,

$$\Gamma^\psi(t-s,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{s,x,u},U_t^{s,x,u})\right], \tag{6.2}$$

so that the estimates hereafter can be extended to the semigroup transitions of the process. We can see that $\Gamma^\psi(T-s,x,u)=F(s,x,u)$.

We consider also the semigroup related to the stopped process:

$$\Gamma_n^\psi(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}},U^{x,u}_{t\wedge\tau_n^{x,u}})\right], \tag{6.3}$$

where $\{\tau_n^{x,u};n\in\mathbb{N}\}$ is the sequence of hitting times defined as

$$\tau_n = \inf\{\tau_{n-1}<t\le T\,;\ X_t\in\partial\mathcal{D}\},\ \text{for } n\ge 1,\quad \tau_0=0,$$

and $\Gamma_0^\psi(t,x,u)=\psi(x,u)$.

When b is a smooth function, the estimates on $\{\Gamma_n^\psi;\ n\ge 1\}$ and Γ^ψ rely on the following PDE result, the proof of which is postponed in the next Subsection 3.1. Let $((x_t^{y,v},u_t^{y,v});\, t\in[0,T])$ be the free Langevin process that verifies

$$\begin{cases} x_t^{y,v} = y + \displaystyle\int_0^t u_s^{y,v}ds,\\ u_t^{y,v} = v + \displaystyle\int_0^t \widetilde{b}(x_s^{y,v},u_s^{y,v})\,ds + \sigma W_t,\end{cases} \tag{6.4}$$

where $\tilde{b}$ is defined in (4.2).

Theorem 6.1. *Assume (H_{PDE}). Assume also that b is a $\mathcal{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^d)$ function. Given two functions $f_0\in L^2(\mathcal{D}\times\mathbb{R}^d)\cap\mathcal{C}_b(\mathcal{D}\times\mathbb{R}^d)$ and $q\in L^2(\Sigma_T^+)\cap\mathcal{C}_b(\Sigma_T^+)$, there exists a unique function $f\in\mathcal{C}_b^{1,1,2}(Q_T)\cap\mathcal{C}((0,T]\times(\overline{\mathcal{D}}\times\mathbb{R}^d\setminus\Sigma^0))\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ which is a solution to*

$$\begin{cases}\partial_t f(t,x,u) - (u\cdot\nabla_x f(t,x,u)) - (b(x,u)\cdot\nabla_u f(t,x,u)) - \dfrac{\sigma^2}{2}\triangle_u f(t,x,u) = 0,\ \textit{for all } (t,x,u)\in Q_T,\\ f(0,x,u) = f_0(x,u),\ \textit{for all } (x,u)\in\mathcal{D}\times\mathbb{R}^d,\\ f(t,x,u) = q(t,x,u),\ \textit{for all } (t,x,u)\in\Sigma_T^+.\end{cases} \tag{6.5}$$

In addition, for $(x_t^{x,u}, u_t^{x,u};\, t \in [0,T])$ *solution to* (6.4) *starting from* $(x,u) \in \mathcal{D} \times \mathbb{R}^d$ *at* $t=0$ *and* $\beta^{x,u} := \inf\{t > 0\,;\, x_t^{x,u} \in \partial\mathcal{D}\}$, *we have*

$$f(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[f_0(x_t^{x,u}, u_t^{x,u})\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right] + \mathbb{E}_{\mathbb{P}}\left[q(t-\beta^{x,u}, x_{\beta^{x,u}}^{x,u}, u_{\beta^{x,u}}^{x,u})\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right]. \tag{6.6}$$

Furthermore, for all $t \in (0,T)$, f *satisfies the inequality:*

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|f\|^2_{L^2(\Sigma_t^-)} + \sigma^2\|\nabla_u f\|^2_{L^2(Q_t)} \leq C_{T,\sigma,\|b\|_{\infty,Lip}}\left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\right) \tag{6.7}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ *is a constant that only depends on* T, σ, *and on the Lipschitz constant in* u, *uniform in* x *of* b, $\|b\|_{\infty,Lip}$.

The proof of this theorem is split in several lemmas and propositions in Appendix 3.1. In Lemma 3.1, we prove the L^p regularity of the solution together with the energy inequality. It is based on the Lions and Magenes' existence theorem stated in 1.3 and on Carrillo's trace existence and Green formula in 1.4. For the inner regularity of the solution, Bouchut's Theorem 1.5 is used to obtain fractional L^p regularity, while bootstrapping techniques are used to increase this regularity to obtain Sobolev estimates to obtain embeddings into continuous spaces in proposition 3.2. Continuity up to the boundary Σ_T^+ is proven using local barrier functions in proposition 3.4 while continuity up to the border Σ_T^- is proven in proposition 3.5 using the Feynman-Kac interpretation (6.6).

Considering the solution f in $\mathcal{C}([0,T]; L^2(\mathcal{D}\times\mathbb{R}^d)) \cap \mathcal{H}(Q_T)$ of (3.1), given by Lemma 3.1, we show its interior regularity and its continuity up to and along $\Sigma_T \setminus \Sigma_T^0$.

From Theorem 6.1, we deduce the following result for $\{\Gamma_n^\psi, n \geq 1\}$:

Corollary 6.2. *Assume (H$_{PDE}$). Assume also that* b *is a* $\mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *function. Then, for all* $\psi \in \mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, *set* $\Gamma_0^\psi = \psi$ *and for all* $n \in \mathbb{N}^*$, Γ_n^ψ *is a function in* $\mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T} \setminus \Sigma^0)$ *and satisfies the PDE*

$$\begin{cases} \partial_t \Gamma_n^\psi(t,x,u) - (u \cdot \nabla_x \Gamma_n^\psi(t,x,u)) - (b(x,u)\cdot\nabla_u\Gamma_n^\psi(t,x,u)) - \dfrac{\sigma^2}{2}\triangle_u\Gamma_n^\psi(t,x,u) = 0, & \textit{for all } (t,x,u) \in Q_T, \\ \Gamma_n^\psi(0,x,u) = \psi(x,u), \ \ \textit{for all } (x,u) \in \mathcal{D}\times\mathbb{R}^d, \\ \Gamma_n^\psi(t,x,u) = \Gamma_{n-1}^\psi(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \ \ \textit{for all } (t,x,u) \in \Sigma_T^+. \end{cases} \tag{6.8}$$

In addition, the set $\{\Gamma_n^\psi, n\geq 1\}$ *belongs to* $L^2((0,T)\times\mathcal{D}; H^1(\mathbb{R}^d))$ *and satisfies the inequality*

$$\|\Gamma_n^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \sigma^2\|\nabla_u\Gamma_n^\psi\|^2_{L^2(Q_t)} + \|\Gamma_n^\psi\|^2_{L^2(\Sigma_t^-)} \leq C_{T,\sigma,\|b\|_{\infty,Lip}}\left(\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_{n-1}^\psi\|^2_{L^2(\Sigma_t^-)}\right) \tag{6.9}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ *is a constant that only depends on* T, b *and* σ.

The proof of this corollary is based on the Theorem 6.1. The unique solution to equation (6.5) with initial condition ψ and boundary condition $\Gamma_{n-1}^\psi(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ when written under its probabilistic interpretation (6.6) is actually equal to Γ_n^ψ defined in (6.3).

Next, by showing the convergence of the Γ_n^ψ to Γ^ψ, we have

Corollary 6.3. *Assume (H$_{PDE}$). Assume also that* b *is a* $\mathcal{C}_b^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ *function. For all* $\psi \in \mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, Γ^ψ *is a function that belongs to* $L^2((0,T)\times\mathcal{D}; H^1(\mathbb{R}^d))$ *and satisfies the inequality:*

$$\|\Gamma^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \sigma^2\|\nabla_u\Gamma^\psi\|^2_{L^2(Q_t)} \leq C_{T,\sigma,\|b\|_{\infty,Lip}}\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}, \ \forall\, t \in (0,T) \tag{6.10}$$

where $C_{T,\sigma,\|b\|_{\infty,Lip}}$ is a positive constant that depends only on T, $\|\nabla_u \cdot b\|_\infty$ and σ. Furthermore, $\Gamma^\psi(t)$ is solution in the sense of distributions of

$$\begin{cases} \partial_t\Gamma^\psi - (u\cdot\nabla_x\Gamma^\psi) - (b(x,u)\cdot\nabla_u\Gamma^\psi) - \dfrac{\sigma^2}{2}\triangle_u\Gamma^\psi = 0, \text{ on } Q_T,\\ \Gamma^\psi(0,x,u) = \psi(x,u), \text{ on } \mathcal{D}\times\mathbb{R}^d,\\ \Gamma^\psi(t,x,u) = \Gamma^\psi(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+. \end{cases} \tag{6.11}$$

The proof of this corollary is given in the Appendix 3.3.

Finally, the following proposition allows to extend the energy estimate (6.10) to the case of drift b satisfying only (H_{PDE}).

Proposition 6.4. *Assume only (H_{PDE}). Then for all $\psi\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$, Γ^ψ defined in (6.1) is a function that belongs to $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and satisfies the inequality (6.10). Furthermore, $\Gamma^\psi(t)$ is solution in the sense of distributions of Equation (6.11).*

Proof. We construct the family $\{b_n, n\in\mathbb{N}\}$ of smooth approximation of b by the following convolution product: for any (x,u) in $\mathcal{D}\times\mathbb{R}^d$,

$$b_n(x,u) = \int_{\mathcal{D}\times\mathbb{R}^d} g_n(u-v)\rho_n(x-y)b(y,v)dydv,$$

where the smoothing kernels g and ρ are as in (3.4) and (3.3), (eventually with the d-product of each kernels to expend the definition to the dimension d). We then define the symetrized extension $\tilde{b}_n$ of b_n on $\mathbb{R}^d\times\mathbb{R}^d$ by

$$\tilde{b}_n\colon (y,v)\in\mathbb{R}^d\times\mathbb{R}^d \mapsto \left(b_n', \operatorname{sign}(y^{(d)})b_n^{(d)}\right)\left((y',|y^{(d)}|),(v',\operatorname{sign}(y^{(d)})v^{(d)})\right), \tag{6.12}$$

and we consider the family of processes $(X_t^n, U_t^n, t\in[0,T])$ and $(Y_t^n, V_t^n, t\in[0,T])$, solution for each fixed n, to the SDEs (1.1) and (1.2), where we have replaced b and $\tilde{b}$ respectively by b_n and $\tilde{b}_n$.

It is classical to observe that b_n inherits from the Lipschitz property of b, with the same constant $\|b\|_{\text{Lip}}$, preserved by the smoothing convolution uniformly in n. Reproducing the arguments in Remark 1.4, we can also deduce that $\tilde{b}_n$ is uniformly Lipschitz on $\mathbb{R}^d\times\mathbb{R}^d$ with constant $2\|b\|_{\text{Lip}}$, and that $\tilde{b}_n$ converges to $\tilde{b}$ uniformly on $\mathbb{R}^d\times\mathbb{R}^d$.

Then the family of processes $(Y_t^n, V_t^n, t\in[0,T])$ belongs in $L^2(\Omega)$ uniformly in time with

$$\begin{aligned} &\sup_{n\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\|Y_t^n\|^2+\|V_t^n\|^2\right]\le C(T,\|b\|_\infty,\|b\|_{\text{Lip}}),\\ &\mathbb{E}\left[\|Y_t^n-Y_s^n\|^2\right]\le C'(T,\|b\|_\infty,\|b\|_{\text{Lip}})|t-s|. \end{aligned}$$

From the relative compactness property, renaming again $(Y_t^n, V_t^n, t\in[0,T])$ a converging sub-sequence with limit $(Y_t^\infty, V_t^\infty, t\in[0,T])$, and from the convergence of $\tilde{b}_n$ to $\tilde{b}$, we check that Y^∞ satisfies (1.2) with drift $\tilde{b}$. By the uniqueness of the solution of (1.2) and also (1.1), we deduce that, for all $t\in[0,T]$,

$$f_n(t,x,u) = \mathbb{E}_{x,u}[\overline{\psi}(Y_t^n,V_t^n)] \xrightarrow[n\to+\infty]{} \mathbb{E}_{x,u}[\overline{\psi}(Y_t,V_t)] = \mathbb{E}_{x,u}[\psi(X_t,U_t)] = \Gamma^\psi(t,x,u)$$

for Γ^ψ defined in (6.1), since the discontinuity points of $(x,u)\mapsto\overline{\psi}(x,u)$ are $\mathcal{P}\circ(Y_t,V_t)^{-1}$-negligible.

Now by applying Corollary 6.3 to f_n and taking the limit with n, we deduce immediately that Γ^ψ is solution to (6.11) in the distribution sense. In particular by Fatou Lemma, the $(\nabla_u f_n, n\ge 0)$ are converging in $L^2(Q_T)$, as n tends to infinity, defining $\nabla_u\overline{\psi}$ as its $L^2(Q_T)$-limit and the Energy inequality (3.17) is preserved. Using the variational formulation of equation (6.11) in the Appendix 3.3, we deduce that Γ^ψ is a $\mathcal{H}(Q_T)$-solution of (6.11) with trace functions $\gamma^\pm(\Gamma^\psi)$ in $L^2(\Sigma^\pm)$. ■

7 Conclusions and perspectives

In this chapter, we have proposed a time discretisation scheme for the specularly reflected Langevin process when the boundary is a hyperplane. We have proven that under the hypotheses ($H_{Langevin}$), (H_{PDE}) and ($H_{Weak\ Error}$) that the weak error produced by this scheme converges to zero at a rate that is at least linear in the time discretisation step.

To obtain the proof, we have used a result from [Bossy and Jabir, 2011] that extends the process on the whole domain, provided that there is a change of drift on the velocity component. The new drift is not even continuous in the most general cases, so we have worked in hypothesis (H_{PDE})-(ii) to consider the cases where under the change, the drift remains continuous. It would be interesting to provide an extension where condition (H_{PDE})-(ii) on the drift would no longer be needed. One possibility would be to obtain more regularity results from the mild equation of the density of the reflected Langevin process defined in [Bossy and Jabir, 2011], another is to obtain regularity results on the hitting times of the position process. Similar results have been obtained in the case of stationary processes in [Geman and Horowitz, 1973].

Another interesting possibility of extending the results is to consider boundaries that are not hyperplanes, by utilising rectification of the boundary techniques such as in [Gilbarg and Trudinger, 2001]. Other results would be to obtain that the weak error actually decreases linearly in the time-step and more importantly to obtain a Richardson-Romberg expansion of the error.

In the next chapter, we shall present some numerical experiments on the proposed discretisation scheme.

Appendices

1 Some recalls

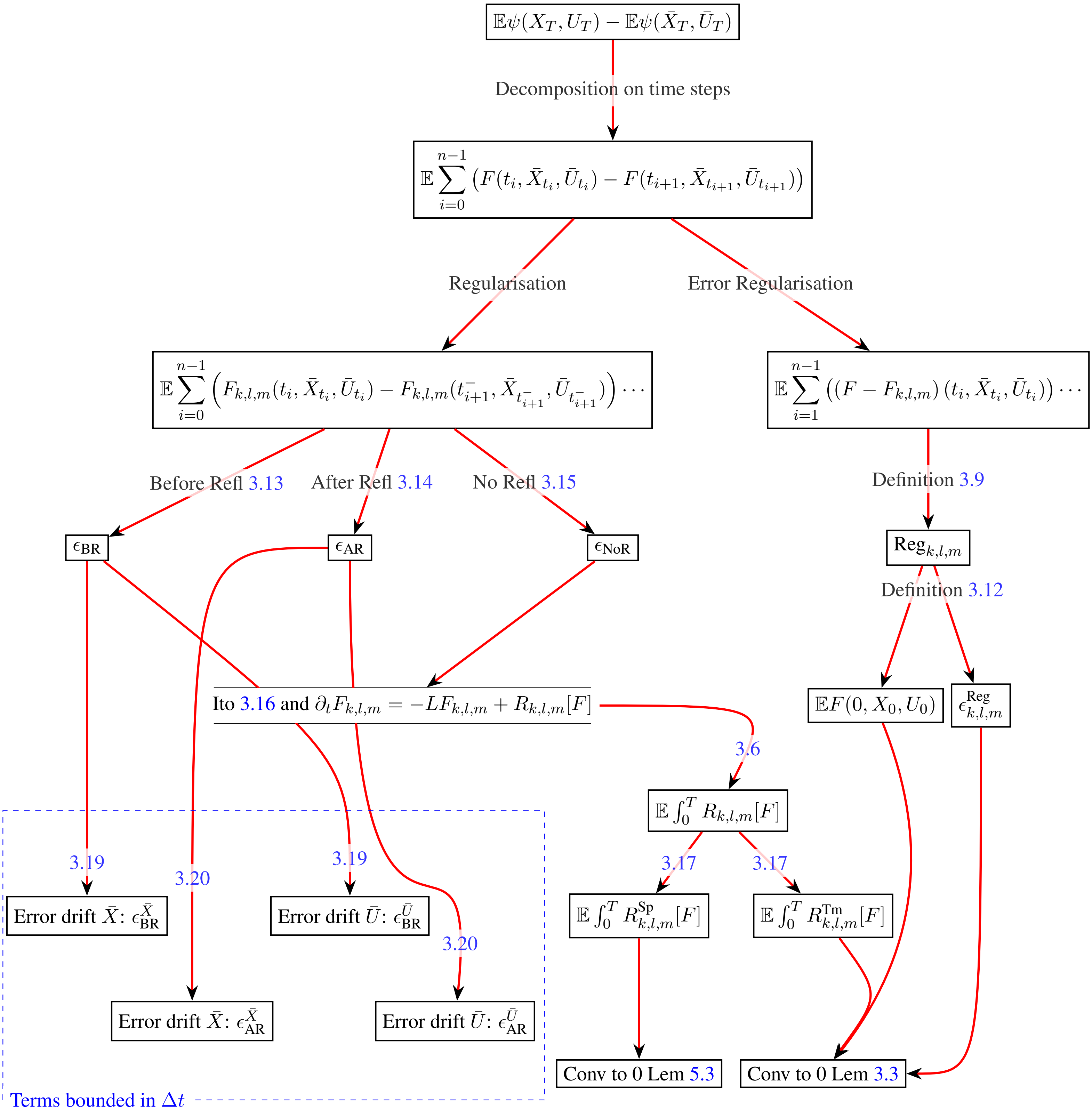


Figure 2: General Schematic of introduced definitions in the main Theorem 1.6

Corollary 1.1 (Rana [Rana, 2005]). *If $\phi \in L^p(\mathbb{R}^d)$ for $p \in [1, +\infty)$ then*

$$\lim_{|\delta| \to 0^+} \int |\phi(z+\delta) - \phi(z)|^p dz = 0.$$

Theorem 1.2 (Tartar [Tartar, 1978], Chapter 4). *Let $\mathcal{V}$ be an open subset of $\mathbb{R}^d$ and $\psi \in L^2(\mathcal{V})$ such that $\nabla_v \psi \in L^2(\mathcal{V})$. Then $\nabla_v(\psi)^+, \nabla_v(\psi)^- \in L^2(\mathcal{V})$ with $\partial_{v_i}(\psi)^+ = \partial_{v_i}\psi \mathbb{1}_{\{\psi \geq 0\}}$ and $\partial_{v_i}(\psi)^- = -\partial_{v_i}\psi \mathbb{1}_{\{\psi \leq 0\}}$.*

Theorem 1.3 (Lions and Magenes [Lions and Magenes, 1972]). *Let E be a Hilbert space with the inner product $(\ ,\ )_E$. Let $F \subset E$ equipped with the norm $|\ |_F$ such that the canonical injection of F into E is continuous. Assume that $A : E \times F \to \mathbb{R}$ is a bilinear application satisfying:*

1. *$\forall\, \psi \in F$, the mapping $A(.,\psi) : E \to \mathbb{R}$ is continuous.*
2. *A is coercive on F that is there exists a constant $c > 0$ such that $A(\psi,\psi) \geq c|\psi|_F^2$, $\forall\, \psi \in F$.*

Then for all linear application $L : F \to \mathbb{R}$, continuous on $(F, |\ |_F)$, there exists $S \in E$ such that $A(S,\psi) = L(\psi)$, $\forall\, \psi \in F$.

Let $\mathcal{T} = \partial_t - u\nabla_x$ be the transport operator and consider the space:

$$\mathcal{Y}(Q_T) = \{\varphi \in \mathcal{H}(Q_T); -\mathcal{T}(\varphi) \in \mathcal{H}'(Q_T)\}.$$

Theorem 1.4 (Carrillo [Carrillo, 1998]). *For any $T > 0$, we have that:*

1. *Let $\varphi \in \mathcal{Y}(Q_T)$. Then:*
   - *φ has a trace $\gamma^+(\varphi) \in L^2(\Sigma_T^+)$ on Σ_T^+ and $\gamma^-(\varphi) \in L^2(\Sigma_T^-)$ on Σ_T^-.*
   - *$\forall t \in [0,T]$, φ has a trace $\varphi(t,\cdot)$ such that the function $t \mapsto \varphi(t,\cdot)$ belongs to $L^2(\mathcal{D} \times \mathbb{R}^d)$.*
2. *For any functions φ, ψ belonging to $\mathcal{Y}(Q_T)$, we have the following Green formula, for any $t \in [0,T]$:*

$$\begin{aligned}(\mathcal{T}(\varphi),\psi)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} - (\mathcal{T}^*(\psi),\varphi)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} &= \int_{\mathcal{D}\times\mathbb{R}^d} \varphi(T,x,u)\psi(T,x,u)\,dxdu \\ - \int_{\mathcal{D}\times\mathbb{R}^d} \varphi(0,x,u)\psi(0,\cdot,\cdot)\,dxdu &- \int_{\Sigma_T^+} (u\cdot n_{\mathcal{D}})\gamma^+(\varphi)(s,x,u)\gamma^+(\psi)(s,x,u)\,d\lambda_\Sigma(s,x,u) \\ - \int_{\Sigma_T^-} (u\cdot n_{\mathcal{D}})\gamma^-(\varphi)(s,x,u)&\gamma^-(\psi)(s,x,u)\,d\lambda_\Sigma(s,x,u)\end{aligned} \tag{1.1}$$

where $\mathcal{T}^* = -\partial_t + u\cdot\nabla_x$, the adjoint of $\mathcal{T}$.

Theorem 1.5 (Bouchut [Bouchut, 2002]). *Let $h \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. Assume that $\phi \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, such that $\nabla_u\phi \in (L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))^d$, satisfies (in the sense of distributions)*

$$\partial_t\phi + (u\cdot\nabla_x\phi) - \frac{\sigma^2}{2}\triangle_u\phi = h, \ \text{ on } \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d. \tag{1.2}$$

Then there exists a positive constant $C(d)$ depending on the dimension such that:

(a) *$\partial_t\phi + (u\cdot\nabla_x\phi)$ and $\triangle_u\phi$ both belong to $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with*

$$\|\partial_t\phi + (u\cdot\nabla_x\phi)\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \frac{\sigma^2}{2}\|\triangle_u\phi\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d)\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)},$$

(b) *$D_x^{2/3}\phi$ and $|\nabla_u D_x^{1/3}\phi|$ belong to $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with*

$$\|\nabla_u D_x^{1/3}\phi\|^2{}_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \|D_x^{2/3}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d)\|h\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}.$$

where for $\alpha\in(0,1)$, D_x^α is the fractional derivative w.r.t. x-variables, defined as the fractional Laplace operator of order α defined as $D_x^\alpha = (-\triangle_x)^{\alpha/2}$.

Lemma 1.6. *Let $T>0$. Consider the mollifying sequence β_k such that* $\mathrm{Supp}(\beta_k)\subset(0,\frac{T}{k})$ *and assume the function $F\colon[0,T]\mapsto\mathbb{R}$ is continuous on $[0,T]$. Then the convolution $F*\beta_k$ converges uniformly towards F on any compact of $(0,T]$.*

Proof. We extend the function F continuously on $\mathbb{R}$ and we denote this continuation as $\widetilde{F}$. By [Brezis, 2010], we have that $\widetilde{F}*\beta_k$ converges uniformly towards $\widetilde{F}$.

Let K_ε be a compact of $(0,T]$ such that the distance $d(0,K_\varepsilon)\geq\varepsilon$. On K_ε, $\widetilde{F}*\beta_k$ converges uniformly towards $\widetilde{F}$. For large enough $k\geq k_\varepsilon$, $\mathrm{Supp}(\beta_k)\cap K_\varepsilon=\emptyset$, and by comparing the supports, for any $t\in K_\varepsilon$, $\widetilde{F}*\beta_k(t)=F*\beta_k(t)$. Let $k\geq k_\varepsilon$:

$$\sup_{t\in K_\varepsilon}\left|\widetilde{F}*\beta_k(t)-\widetilde{F}(t)\right| = \sup_{t\in K_\varepsilon}|F*\beta_k(t)-F(t)| \xrightarrow{k\to\infty} 0$$

and we conclude. ∎

2 Complement to Lemma 2.2 about the density of the discretized free Langevin process

Lemma 2.1. *The transition density of the discretized version of the free Langevin process*

$$\begin{cases} \bar{Z}_t = x + tu + \sigma\displaystyle\int_0^t W_{\eta(s)}\,ds \\ \bar{V}_t = u + \sigma W_t \end{cases} \tag{2.1}$$

is a Gaussian transition density

$$\bar{p}^L(0;x,u;t;\xi,\zeta) = p_{\mathcal{N}(0,\Sigma_{t,\eta(t),\Delta t})}(\xi-(x+tu),\zeta-u)$$

where $p_{\mathcal{N}(0,\Gamma)}$ denotes the centered Gaussian density with covariance Γ, and

$$\Sigma_{t,\eta(t),\Delta t} = \sigma^2\begin{bmatrix} t\eta(t)(t-\eta(t)-\Delta t)+\frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6} & t\eta(t)-\frac{\eta(t)(\eta(t)+\Delta t)}{2} \\ t\eta(t)-\frac{\eta(t)(\eta(t)+\Delta t)}{2} & t \end{bmatrix}$$

is degenerate in its first coordinate when $t<\Delta t$.

Proof. We have that $\mathbb{E}\bar{Z}_t = x+tu$ and $\mathbb{E}\bar{V}_t = u$. Also $\mathbb{V}\mathrm{ar}\left[\bar{V}_t\right]=\sigma^2 t$ and

$$\begin{aligned}
&\int_0^t W_{\eta(s)}\,ds = \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} W_{t_i}(t_{i+1}-t_i) + (t-\eta(t))W_{\eta(t)} \\
&= \left(t_{\lfloor\frac{t}{\Delta t}\rfloor}W_{t_{\lfloor\frac{t}{\Delta t}\rfloor}} - t_0W_0\right) - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}}-W_{t_i}\right) + (t-\eta(t))W_{\eta(t)} \\
&= \eta(t)W_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}}-W_{t_i}\right) + (t-\eta(t))W_{\eta(t)} = tW_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}}-W_{t_i}\right),
\end{aligned}$$

and by computing the variance of the previous sum:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[\int_0^t W_{\eta(s)}\,ds\right] &= \mathbb{V}\mathrm{ar}\left[tW_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] \\
&= t^2\mathbb{V}\mathrm{ar}\left[W_{\eta(t)}\right] + \mathbb{V}\mathrm{ar}\left[\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] - 2t\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\mathbb{C}\mathrm{ov}\left[W_{\eta(t)}, W_{t_{i+1}} - W_{t_i}\right] \\
&= t^2\eta(t) + \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}^2\Delta t - 2t\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\Delta t \\
&= t^2\eta(t) + (\Delta t)^3\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1}(i+1)^2 - 2t(\Delta t)^2\sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1}(i+1) \\
&= t^2\eta(t) + (\Delta t)^3\frac{\lfloor\frac{t}{\Delta t}\rfloor\left(\lfloor\frac{t}{\Delta t}\rfloor+1\right)\left(2\lfloor\frac{t}{\Delta t}\rfloor+1\right)}{6} - 2t(\Delta t)^2\frac{\lfloor\frac{t}{\Delta t}\rfloor\left(\lfloor\frac{t}{\Delta t}\rfloor+1\right)}{2} \\
&= t\eta(t)(t-\eta(t)-\Delta t) + \frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6}.
\end{aligned}$$

Concerning the covariance:

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\left[W_t, \int_0^t W_{\eta(s)}\,ds\right] &= \mathbb{C}\mathrm{ov}\left[W_t, tW_{\eta(t)} - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\left(W_{t_{i+1}} - W_{t_i}\right)\right] \\
&= t\eta(t) - \sum_{i=0}^{\lfloor\frac{t}{\Delta t}\rfloor-1} t_{i+1}\Delta t = t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2}.
\end{aligned}$$

Finally, we obtain that:

$$\begin{bmatrix}\bar{Z}_t^{x,u}\\ \bar{V}_t^{x,u}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}x+tu\\ u\end{bmatrix}, \sigma^2\begin{bmatrix} t\eta(t)(t-\eta(t)-\Delta t) + \frac{\eta(t)(\eta(t)+\Delta t)(2\eta(t)+\Delta t)}{6} & t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} \\ t\eta(t) - \frac{\eta(t)(\eta(t)+\Delta t)}{2} & t\end{bmatrix}\right).$$

Let $t > \Delta t$ and $\Sigma_{t,\eta(t),\Delta t}$ denotes the above covariance matrix. When $t \neq \eta(t)$, we consider the index $k \in \mathbb{N}^*$ such that $\eta(t) = k\Delta t$ and $\varepsilon \in (0,1)$ such that $t = \varepsilon + k\Delta t$. It can be easily seen that:

$$\det(\Sigma_{t,\eta(t),\Delta t}) = \frac{1}{12}k\Delta t\left(12\varepsilon^3 + 12(k-1)\Delta t\varepsilon + 2(k-1)(2k-1)(\Delta t)^2\varepsilon + k(k-1)(\Delta t)^3\right).$$

So it can be seen that $\det(\Sigma_{t,\eta(t),\Delta t}) > 0$. Then, for $t = \eta(t) > \Delta t$ then:

$$\det(\Sigma_{t,\eta(t),\Delta t}) = \frac{\eta(t)^2}{12}\left(\eta(t)^2 - (\Delta t)^2\right) > 0.$$

Then, for any $t > \Delta t$, we have that the pdf of the r.v. $(\bar{Z}_t^{x,u}, \bar{V}_t^{x,u})$ is:

$$\bar{p}^L(0;x,u;t;\xi,\zeta) = \frac{1}{2\pi\sqrt{\det(\Sigma_{t,\eta(t),\Delta t})}}\exp\left(-\frac{1}{2}\begin{bmatrix}\xi-(x+tu)\\ \zeta-u\end{bmatrix}^T \Sigma_{t,\eta(t),\Delta t}\begin{bmatrix}\xi-(x+tu)\\ \zeta-u\end{bmatrix}\right).$$

When $t \leq \Delta t$, the position process is a degenerate random variable and the pdf becomes:

$$\bar{p}^L(0;x,u;\xi,\zeta) = \delta(\xi-(x+tu))\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(\zeta-u)^2}{2\sigma^2 t}\right)$$

where δ is the Dirac delta distribution. ■

3 Some complements to Section 6

For the sake of completeness, we present in this appendix the proofs of the section 6.

3.1 Proof of Theorem 6.1

We assume that the drift b is a $\mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ function.

We consider the inputs (f_0, q) and assume the following

$$(H_{f_0,q}): \qquad f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d) \cap \mathcal{C}_b(\mathcal{D} \times \mathbb{R}^d) \text{ and } q \in L^2(\Sigma_T^+) \cap \mathcal{C}_b(\Sigma_T^+).$$

As a preliminary for the proof of Theorem 6.1, let us recall a more classical existence result for equation (6.5), issued from the application of Lions and Magenes Theorem 1.3.

Lemma 3.1. *Assume (H_{PDE}). Given two functions $f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d)$ and $q \in L^2(\Sigma_T^+)$, there exists a unique function f in $\mathcal{C}([0,T]; L^2(\mathcal{D} \times \mathbb{R}^d)) \cap \mathcal{H}(Q_T)$ admitting a trace $\gamma(f) \in L^2(\Sigma_T)$ along the boundary Σ_T, satisfying equation (6.5) in the sense that*

$$\begin{aligned}
&\partial_t f - (u \cdot \nabla_x f) - (b(x,u) \cdot \nabla_u f) - \frac{\sigma^2}{2} \triangle_u f = 0, \text{ in } \mathcal{H}'(Q_T),\\
&f(t=0,x,u) = f_0(x,u), \text{ on } \mathcal{D} \times \mathbb{R}^d,\\
&\gamma(f)(t,x,u) = q(t,x,u), \text{ on } \Sigma_T^+.
\end{aligned} \tag{3.1}$$

In particular, for all $t \in (0,T)$,

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \sigma^2 \|\nabla_u f\|^2_{L^2(Q_t)} + \|\gamma(f)\|^2_{L^2(\Sigma_t^-)} \leq C_{T,\|\nabla_u \cdot b\|_\infty,\sigma} \left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)} \right) \tag{3.2}$$

where $C_{T,\|\nabla_u \cdot b\|_\infty,\sigma}$ is a positive constant depending on T, b and σ.

Proof. **Step 1: Construction of a solution in $\mathcal{H}(Q_T)$**

Let λ be a real to be defined later on and the functions $\bar{f}\colon (t,x,u) \in Q_T \mapsto \exp(-\lambda t) f(t,x,u)$ and $\bar{q}\colon (t,x,u) \in \Sigma_T^+ \mapsto \exp(-\lambda t) q(t,x,u)$. Then (3.1) becomes:

$$\begin{aligned}
&\partial_t \bar{f} - (u \cdot \nabla_x \bar{f}) - (b(x,u) \cdot \nabla_u \bar{f}) - \frac{\sigma^2}{2} \triangle_u \bar{f} + \lambda \bar{f} = 0, \text{ in } \mathcal{H}'(Q_T),\\
&\bar{f}(t=0,x,u) = f_0(x,u), \text{ on } \mathcal{D} \times \mathbb{R}^d,\\
&\gamma(\bar{f})(t,x,u) = \bar{q}(t,x,u), \text{ on } \Sigma_T^+.
\end{aligned} \tag{3.3}$$

In order to apply Theorem (1.3), the space E is identified as $\mathcal{H}(Q_T)$ considered with its norm. Also we define the space $F = \{\psi \in \mathcal{C}_c^\infty(\overline{Q}_T; \mathbb{R}),$ s.t. $\psi = 0$ on $\{T\} \times \mathcal{D} \times \mathbb{R}^d$ and on $\overline{\Sigma_T^-}\}$ together with its norm:

$$|\psi|_F^2 = \|\psi\|^2_{\mathcal{H}(Q_T)} + \|\psi\|^2_{L^2(\Sigma_T^+)}$$

which shows that the canonical injection from F into $\mathcal{H}(Q_T)$ is continuous.

The variational form of (3.3) is written as: for $\psi \in F$,

$$\begin{aligned}
&\int_{Q_T} \psi \partial_t \bar{f} - \int_{Q_T} \psi(u \cdot \nabla_x \bar{f}) - \int_{Q_T} \psi(b(x,u) \cdot \nabla_u \bar{f}) - \frac{\sigma^2}{2} \int_{Q_T} \psi \triangle_u \bar{f} + \lambda \int_{Q_T} \psi \bar{f} = 0 \\
\iff \quad & - \int_{Q_T} \bar{f} \partial_t \psi + \int_{Q_T} \bar{f}(u \cdot \nabla_x \psi) - \int_{Q_T} \psi(b(x,u) \cdot \nabla_u \bar{f}) + \frac{\sigma^2}{2} \int_{Q_T} \nabla_u \psi \cdot \nabla_u \bar{f} + \lambda \int_{Q_T} \psi \bar{f} \\
&= \int_{\Sigma_T} (u \cdot n_{\mathcal{D}}) \psi \bar{f} - \int_{\mathcal{D} \times \mathbb{R}^d} \psi(T, \cdot, \cdot) \bar{f} + \int_{\mathcal{D} \times \mathbb{R}^d} \psi(0, \cdot, \cdot) f_0 \\
&= - \int_{\Sigma_T^-} |(u \cdot n_{\mathcal{D}})| \psi \bar{f} + \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \psi \bar{q} - \int_{\mathcal{D} \times \mathbb{R}^d} \psi(T, \cdot, \cdot) \bar{f} + \int_{\mathcal{D} \times \mathbb{R}^d} \psi(0, \cdot, \cdot) f_0 \\
&= \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \psi \bar{q} + \int_{\mathcal{D} \times \mathbb{R}^d} \psi(0, \cdot, \cdot) f_0
\end{aligned}$$

which allows to identify the bilinear form $A\colon (\varphi, \psi) \in (\mathcal{H}(Q_T) \times F) \mapsto A(\varphi, \psi)$ as:

$$A(\varphi, \psi) = - \int_{Q_T} \varphi \partial_t \psi + \int_{Q_T} \varphi(u \cdot \nabla_x \psi) - \int_{Q_T} (b(x,u) \cdot \nabla_u \varphi) \psi + \frac{\sigma^2}{2} \int_{Q_T} \nabla_u \varphi \cdot \nabla_u \psi + \lambda \int_{Q_T} \varphi \psi$$

and the linear form $L\colon \psi \in F \mapsto L(\psi)$:

$$L(\psi) = \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \bar{q} \psi + \int_{\mathcal{D} \times \mathbb{R}^d} f_0(0, \cdot, \cdot) \psi.$$

So the shorthand version of the variational form of (3.3) is:

$$A(\bar{f}, \psi) = L(\psi). \tag{3.4}$$

It is clear that the mapping $A(\cdot, \psi)$ from $\mathcal{H}(Q_T)$ into $\mathbb{R}$ is continuous for any ψ in F. Concerning the coercivity, for any ψ in F:

$$\begin{aligned}
A(\psi, \psi) &= - \int_{Q_T} \psi \partial_t \psi + \int_{Q_T} \psi(u \cdot \nabla_x \psi) - \int_{Q_T} (b(x,u) \cdot \nabla_u \psi) \psi + \frac{\sigma^2}{2} \int_{Q_T} \nabla_u \psi \cdot \nabla_u \psi + \lambda \int_{Q_T} \psi^2 \\
&= -\frac{1}{2} \|\psi(T, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \|\psi(0, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \int_{Q_T} u \cdot \nabla_x \psi^2 - \frac{1}{2} \int_{Q_T} b(x,u) \cdot \nabla_u \psi^2 \\
&\quad + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} + \lambda \|\psi\|^2_{L^2(Q_T)} \\
&= \frac{1}{2} \|\psi(0, \cdot, \cdot)\|^2_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \frac{1}{2} \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}})| \psi^2 + \frac{1}{2} \int_{Q_T} (\nabla_u \cdot b(x,u)) \psi^2 + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} + \lambda \|\psi\|^2_{L^2(Q_T)} \\
&\geq \frac{1}{2} \|\psi\|^2_{L^2(\Sigma_T^+)} + \left(\lambda - \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)} \right) \|\psi\|^2_{L^2(Q_T)} + \frac{\sigma^2}{2} \|\nabla_u \psi\|^2_{L^2(Q_T)} \\
&\geq \min \left(\lambda - \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)}, \frac{\sigma^2}{2}, \frac{1}{2} \right) |\psi|^2_F.
\end{aligned}$$

By choosing $\lambda > \frac{1}{2} \|\nabla_u \cdot b(x,u)\|_{L^\infty(Q_T)}$, A becomes a coercive application on $F \times F$ and, as such, by Theorem (1.3), there exists $\bar{f}$ in $\mathcal{H}(Q_t)$ such that for any ψ in F, the equation (3.4) is satisfied. Multiplying this function by $\exp(\lambda t)$ gives the desired result.

Step 2: Existence of the trace on Σ_T and proof of energy inequality

Consider now the transport operator $\mathcal{T} = \partial_t - u \cdot \nabla_x$ and the spaces:

$$\mathcal{Y}(Q_T) = \{\varphi \in \mathcal{H}(Q_T); -\mathcal{T}(\varphi) \in \mathcal{H}'(Q_T)\}$$

and

$$\mathcal{V}(Q_T) = \{\psi \in \mathcal{H}(Q_T); \psi \text{ has traces } \gamma(\psi^{\pm}) \text{ on } \Sigma_T^{\pm}, \gamma(\psi^{\pm}) \in L^2(\Sigma_T^{\pm})\}.$$

We shall show that $f \in \mathcal{Y}(Q_T)$. Let $\varphi \in \mathcal{C}_c^{\infty}(Q_T)$, then:

$$\begin{aligned}
\left| \int_{Q_T} \mathcal{T}(f)\varphi \right| &= \left| \int_{Q_T} -\varphi b \cdot \nabla_u f - \frac{\sigma^2}{2} \int_{Q_T} \triangle_u f \varphi \right| \\
&\leq \|b\|_{L^{\infty}(\mathcal{D}\times\mathbb{R}^d)} \int_{Q_T} |\nabla_u f||\varphi| + \left| \int_{Q_T} \nabla_u f \nabla_u \varphi \right| \\
&\leq \max\left\{ \|b\|_{L^{\infty}(\mathcal{D}\times\mathbb{R}^d)}, 1 \right\} \|f\|_{\mathcal{H}(Q_T)} \|\varphi\|_{\mathcal{H}(Q_T)}.
\end{aligned}$$

This means that $f \in \mathcal{Y}(Q_T)$, so by [Carrillo, 1998], f admits a trace on the border of the domain $\mathcal{D}$ and $f \in \mathcal{V}(Q_t)$, and the Green formula (1.4) can be applied. Equation (3.1) can be rewritten as:

$$\mathcal{T}(f)(t,x,u) - b(x,u) \cdot \nabla_u f(t,x,u) - \frac{\sigma^2}{2} \triangle_u f = 0$$

and by multiplying with f and integrating over Q_T, we obtain that:

$$\begin{aligned}
&(\mathcal{T}(f), f)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} - \int_{Q_T} (b \cdot \nabla_u f) f - \frac{\sigma^2}{2} \int_{Q_T} f \triangle_u f = 0 \\
\Longleftrightarrow\ &(\mathcal{T}^*(f), f)_{\mathcal{H}'(Q_T),\mathcal{H}(Q_T)} + \int_{\mathcal{D}\times\mathbb{R}^d} f^2(T,x,u)\, dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} f_0^2 \\
&- \int_{\Sigma_T^+} (u \cdot n_{\mathcal{D}}) q^2 - \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}) \gamma^-(f)^2 - \int_{Q_T} (b \cdot \nabla_u f) f - \frac{\sigma^2}{2} \int_{Q_T} f \triangle_u f = 0
\end{aligned}$$

Since $\mathcal{T}^* = -\mathcal{T}$, we add the two previous equations to obtain that:

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} f^2(T,x,u)\, dxdu - \int_{\mathcal{D}\times\mathbb{R}^d} f_0^2 - \int_{\Sigma_T^+} (u \cdot n_{\mathcal{D}}) q^2 \\
&- \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}) \gamma^-(f)^2 - \int_{Q_T} 2(b \cdot \nabla_u f) f - \sigma^2 \int_{Q_T} f \triangle_u f = 0
\end{aligned}$$

As T is arbitrary, this also writes for any $t \leq T$,

$$\begin{aligned}
&\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \int_{\Sigma_t^-} |u \cdot n_{\mathcal{D}}| (\gamma(f)^-)^2 + \int_{Q_t} (\nabla_u \cdot b) f^2 + \sigma^2 \|\nabla_u f\|^2_{L^2(Q_t,\mathbb{R}^d)} \\
&= \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \int_{\Sigma_t^+} |u \cdot n_{\mathcal{D}}| (\gamma(f)^+)^2
\end{aligned}$$

and

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\gamma(f)\|^2_{L^2(\Sigma_t^-)} + \sigma^2 \|\nabla_u f\|^2_{L^2(Q_t)} = -\int_{Q_t} (\nabla_u \cdot b) f^2 + \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}.$$

From this, one has the inequality:

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \leq \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)} + \|\nabla_u \cdot b\|_{L^{\infty}(Q_t)} \int_0^t ds\, \|f(s)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}$$

So by Grownall's lemma:

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \leq \left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\right) \exp\left(\frac{1}{2}\|\nabla_u \cdot b\|_{L^\infty(Q_t)}\, t\right)$$

$$\text{and} \quad \|f\|^2_{L^2(Q_t)} \leq t\left(\|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)}\right) \exp\left(\frac{1}{2}\|\nabla_u \cdot b\|_{L^\infty(Q_t)}\, t\right)$$

which when plugged in the previous equality, allows to obtain (3.2) with

$$C_{T,\|\nabla_u\cdot b\|_\infty,\sigma} = 1 + T\,\|\nabla_u \cdot b\|_{L^\infty(Q_T)} \exp\left(\|\nabla_u \cdot b\|_{L^\infty(Q_T)}\, T\right).$$

The uniqueness of the solution is obtained from the energy inequality and linearity of the equation. ■

Proposition 3.2 (Interior regularity). *Under $(H_{f_0,q})$, the unique solution f of (3.1) belongs to $\mathcal{C}^{1,1,2}(Q_T)$.*

Proof. To prove this proposition, it is sufficient to show that, for all $z_0 := (t_0, x_0, u_0)$ in Q_T, there exists $r > 0$ such that f belongs to $\mathcal{C}^{1,1,2}(B_{z_0}(r))$ where $B_{z_0}(r) \subset Q_T$ is the open ball centred at z_0 of radius r. To this end, we use the Sobolev embeddings (see e.g. [Brezis, 2010], Corollary 9.15): for $m = \lfloor d/2 \rfloor + 2 - \lfloor 1 - (d/2 - \lfloor d/2 \rfloor)\rfloor$, we have[2]

$$W^{2,2}((0,T)) \subset \mathcal{C}^1([0,T]), \quad W^{m,2}(B_{x_0}(r)) \subset \mathcal{C}^1(\overline{B_{x_0}(r)}), \quad W^{m+1,2}(B_{u_0}(r)) \subset \mathcal{C}^2(\overline{B_{u_0}(r)}).$$

We thus first prove that for some $r > 0$,

$$\|\partial_t^2 f\|_{L^2(B_{z_0}(r))} + \sum_{\eta\in\mathbb{N}^d;|\eta|\leq m} \|D_x^\eta f\|_{L^2(B_{z_0}(r))} + \sum_{\kappa\in\mathbb{N}^d;|\kappa|\leq m+1} \|D_u^\kappa f\|_{L^2(B_{z_0}(r))} < +\infty, \tag{3.5}$$

where D_x^η and D_u^κ refer to the differential operators given by

$$\begin{aligned} D_x^\eta f =& \partial_{x_1}^{\eta_1}\partial_{x_2}^{\eta_2}\cdots\partial_{x_d}^{\eta_d} f, \text{ for } \eta = (\eta_1, \eta_2, \cdots, \eta_d) \in \mathbb{N}^d, \\ D_u^\kappa f =& \partial_{u_1}^{\kappa_1}\partial_{u_2}^{\kappa_2}\cdots\partial_{u_d}^{\kappa_d} f, \text{ for } \kappa = (\kappa_1, \kappa_2, \cdots, \kappa_d) \in \mathbb{N}^d. \end{aligned}$$

Since b is assumed to be a $\mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ function here, we can iterate the whole argument and prove (3.5) for higher order of Sobolev derivatives to conclude that f belongs to $\mathcal{C}^{1,1,2}(B_{z_0}(r))$.

The proof of (3.5), is based on a bootstrap argument that uses the regularity results (in fractional Sobolev spaces) obtained in Bouchut [Bouchut, 2002] for the solution to kinetic equation (see Theorem 1.5).

Step 1. Let us start with the regularity along the (x,u)-variables. We proceed by induction on a truncated version of f.

For any $r_0 > 0$ such that $B_{z_0}(r_0) \subsetneq Q_T$, we denote by $\beta_{r_0} : Q_T \to [0,1]$, a $\mathcal{C}_c^\infty(Q_T)$-cutoff function such that

$$\begin{cases} \beta_{r_0} = 1 \text{ on } \overline{B_{z_0}(\frac{r_0}{2})}, \\ \beta_{r_0} = 0 \text{ on } Q_T \setminus B_{z_0}(r_0). \end{cases}$$

We further assume that there exists a constant C depending on r_0 such that

$$\sum_{\eta\in\mathbb{N}^d;|\eta|\leq m+1;\, \beta\in\mathbb{N}^d;|\beta|\leq m+2} \|\partial_t^2 D_x^\eta D_u^\beta \beta_{r_0}\|_{L^\infty(Q_T)} \leq C.$$

[2] For $\lfloor x \rfloor$ the nearest integer lower than $x \in \mathbb{R}^+$.

Starting from $f \in L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ given in Lemma 3.1, the truncated function $f_{r_0} := \beta_{r_0} f$ satisfies, in the sense of distributions,

$$\begin{cases} \partial_t f_{r_0} - (u\cdot\nabla_x f_{r_0}) - \dfrac{\sigma^2}{2}\triangle_u f_{r_0} = \Gamma_{r_0} f + (\Psi_{r_0}\cdot\nabla_u f), \text{ on } Q_T, \\ f_{r_0}|_{t=0} = 0, \text{ on } \mathcal{D}\times\mathbb{R}^d, \\ \gamma^{\pm}(f_{r_0}) = 0, \text{ on } \Sigma_T^{\pm}, \end{cases}$$

with $\Gamma_{r_0} := \partial_t\beta_{r_0} - (u\cdot\nabla_u\beta_{r_0}) - \frac{\sigma^2}{2}\triangle_u\beta_{r_0}$ and $\Psi_{r_0} := -\sigma^2\nabla_u\beta_{r_0} + (\beta_{r_0} b)$. Extending f_{r_0}, $\Gamma_{r_0} f$ and $(\Psi_{r_0}\cdot\nabla_u f)$ on the whole space $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ by 0 outside $B_{z_0}(r_0)$, one has

$$\partial_t f_{r_0} - (u\cdot\nabla_x f_{r_0}) - \frac{\sigma^2}{2}\triangle_u f_{r_0} = g_{r_0}, \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))' \tag{3.6}$$

where $g_{r_0} := \Gamma_{r_0} f + (\Psi_{r_0}\cdot\nabla_u f)$. Let us now recall Theorem 1.5 (and its proof) in [Bouchut, 2002]: for $\alpha\in(0,1)$, we further denote by D_x^α the fractional derivative w.r.t. x-variables, defined as the fractional Laplace operator of order α

$$D_x^\alpha = (-\triangle_x)^{\alpha/2}.$$

Since $\Gamma_{r_0} f$ and $(\Psi_{r_0}\cdot\nabla_u f)$ are in $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, Theorem 1.5-(b) implies that $D_x^{2/3} f_{r_0}$, $|\nabla_u D_x^{1/3} f_{r_0}|$, and $\triangle_u f_{r_0}$ are in $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. As $\beta_{r_0} = 1$ on $B_{z_0}(\frac{r_0}{2})$, this particularly ensures that

$$\begin{aligned} &\|D_x^{2/3} f\|_{L^2(B_{z_0}(\frac{r_0}{2}))} = \|D_x^{2/3} f_{r_0}\|_{L^2(B_{z_0}(\frac{r_0}{2}))} \le \|D_x^{2/3} f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} < +\infty, \\ &\|\nabla_u D_x^{1/3} f\|_{L^2(B_{z_0}(\frac{r_0}{2}))} = \|\nabla_u D_x^{1/3} f_{r_0}\|_{L^2(B_{z_0}(\frac{r_0}{2}))} \le \|\nabla_u D_x^{1/3} f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} < +\infty. \end{aligned}$$

By setting $r_1 := \frac{r_0}{2}$ and $f_{r_1} := \beta_{r_1} f$, it follows that $D_x^{1/3} f_{r_1} \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ (since[3] $\|D_x^{1/3} f_{r_1}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \le \|f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}\|D_x^{2/3} f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}$.) Furthermore, as we are dealing with L^2 norm, the fractional Sobolev space H^α, $0<\alpha<1$ and the fractional Laplacian operator D^α are connected and (see [Nezza *et al.*, 2012], Proposition 3.6), $\|f\|_{H^s} = C\|D^\alpha f\|_{L^2}$ for C a dimensional constant. Moreover, as g_{r1} is the product of $\mathcal{C}_c^\infty$ functions with f_{r1} and $\nabla_u f_{r1}$, we can apply the Lemma 5.3 in [Nezza *et al.*, 2012], to get

$$\|D_x^{1/3} g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} = C\|g_{r1}\|_{H^{1/3}(B_{z_0}(r_1))} \le C'\|f\|_{H^{1/3}(B_{z_0}(r_1))} + C'\|\nabla_u f\|_{H^{1/3}(B_{z_0}(r_1))} < \infty.$$

Applying the differential operator $D_x^{1/3}$ to (3.6), one can check that $D_x^{1/3} f_{r_1}$ satisfies

$$\partial_t D_x^{1/3} f_{r_1} - (u\cdot\nabla_x D_x^{1/3} f_{r_1}) - \frac{\sigma^2}{2}\triangle_u D_x^{1/3} f_{r_1} = D_x^{1/3} g_{r_1}, \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'. \tag{3.7}$$

From Theorem 1.5-(b) again, we obtain that $|\nabla_x f_{r_1}| \le C|D_x^{2/3}(D_x^{1/3} f_{r_1})| \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$, and $|\nabla_u D_x^{2/3} f_{r_1}| \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. Therefore, $|\nabla_x f| \in L^2(B_{z_0}(\frac{r_1}{2}))$. Applying again $D_x^{1/3}$ to (3.7), applying Theorem 1.5-(b) a third time, one can also deduce that $|\nabla_u\nabla_x f|$ is in $L^2(B_{z_0}(\frac{r_0}{2^3}))$.

We obtain the regularity w.r.t. u by applying the differential operator ∂_{u_i} to Eq. (3.6). Hence $\partial_{u_i} f_{r_1}$ satisfies

$$\partial_t\partial_{u_i} f_{r_1} - (u\cdot\nabla_x\partial_{u_i} f_{r_1}) - \frac{\sigma^2}{2}\triangle_u\partial_{u_i} f_{r_1} = \partial_{u_i} g_{r_1} + \partial_{x_i} f_{r_1}, \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))', \tag{3.8}$$

where

$$\partial_{u_i} g_{r_1} = (\partial_{u_i}\Gamma_{r_1}) f + \Gamma_{r_1}\partial_{u_i} f + (\Psi_{r_1}\cdot\nabla_u\partial_{u_i} f) + (\partial_{u_i}\Psi_{r_1}\cdot\nabla_u f).$$

[3] This can be shown by applying a Cauchy-Schwarz inequality in the alternative definition of the fractional derivative in L^2 via Fourier transform, see e.g. [Nezza *et al.*, 2012].

Theorem 1.5-(a) ensures that $\|\triangle_u f_{r_0}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} < +\infty$. As f_{r_0} has a compact support, standard arguments give that

$$\sum_{1\leq i,j\leq d} \|\partial^2_{u_i,u_j} f_{r_0}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} = \|\triangle_u f_{r_0}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} < +\infty$$

and thus

$$\sum_{1\leq i,j\leq d} \|\partial^2_{u_i,u_j} f\|^2_{L^2(B_{z_0}(r_1))} = \sum_{1\leq i,j\leq d} \|\partial^2_{u_i,u_j} f_{r_0}\|_{L^2(B_{z_0}(r_1))} < +\infty.$$

Now we set $h = \partial_{u_i} g_{r_1} + \partial_{x_i} f_{r_1}$ with $\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq \|\nabla_u g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \|\nabla_x f_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} < +\infty$, since

$$\|\nabla_u g_{r_1}\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C\big(\|f\|_{L^2(B_{z_0}(r_1))} + \|\nabla_u f\|_{L^2(B_{z_0}(r_1))} + \sum_{1\leq i,j\leq d} \|\partial^2_{u_i,u_j} f\|^2_{L^2(B_{z_0}(r_1))}\big) < +\infty.$$

Theorem 1.5-(a) ensures that $|\nabla_u(\triangle_u f_{r_1})| \in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ and hence that $|\triangle_u \nabla_u f| \in L^2(B_{z_0}(\frac{r_1}{2}))$.

We sum up the estimations we have obtained as

$$\|\nabla_x f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \|\nabla_x \nabla_u f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \|\triangle_u \nabla_u f\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} < +\infty. \tag{3.9}$$

We extend (3.9) to higher order differentials through the following induction argument: we have proved that for $N = 1$,

$D^\eta_x f,\ |\nabla_u D^\eta_x f|,\ |\nabla_x D^{\eta'}_u f|,\ |\nabla_u D^\eta_u f|$ are all in $L^2(B_{z_0}(R_N))$, for all $\eta \in \mathbb{N}^d$ such that $1 \leq |\eta| \leq N$,

with $R_N = r_0/2^{3N}$ and $\eta' \in \mathbb{N}^d$ is such that $|\eta'| = |\eta| - 1$.

Starting from the induction assumption that $\|D^\eta_x f\|_{L^2(B_{z_0}(R_N))} + \|\nabla_u D^\eta_x f\|_{L^2(B_{z_0}(R_N))} < +\infty$, for $|\eta| \leq N$, we have that $D^\eta_x f_{R_N}$ satisfies

$$\partial_t D^\eta_x f_{R_N} - (u \cdot \nabla_x D^\eta_x f_{R_N}) - \frac{\sigma^2}{2} \triangle_u D^\eta_x f_{R_N} = D^\eta_x g_{R_N},\ \text{ in } (\mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'.$$

Applying three times Theorem 1.5-(b), we deduce as before that $|\nabla_x D^\eta_x f|$ and $|\nabla_x \nabla_u D^\eta_x f|$ are in $L^2(B_{z_0}(\frac{R_N}{2^3}))$.

Now, from the induction assumption $\|\nabla_u D^\eta_u f\|_{L^2(B_{z_0}(R_N))} + \|\nabla_x D^{\eta'}_u f\|_{L^2(B_{z_0}(R_N))} < +\infty$, for η and η', $|\eta| \leq N$, we have that $D^\eta_u f_{R_N}$ satisfies

$$\partial_t D^\eta_u f_{R_N} - (u \cdot \nabla_x D^\eta_u f_{R_N}) - \frac{\sigma^2}{2} \triangle_u D^\eta_u f_{R_N} = D^\eta_u g_{R_N} + \left(D^\eta_u(u \cdot \nabla_x f_{R_N}) - (u \cdot \nabla_x D^\eta_u f_{R_N})\right),$$

in $(\mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))'$. Since

$$\|D^\eta_u(u \cdot \nabla_x f_{R_N}) - (u \cdot \nabla_x D^\eta_u f_{R_N})\| \leq \sum_{\eta';|\eta'|=N-1} \|\nabla_x D^{\eta'}_u f\|_{L^2(B_{z_0}(R_N))} < +\infty,$$

applying Theorem 1.5-(a), we deduce as before that $\triangle_u D^\eta_u f \in L^2(B_{z_0}(\frac{R_N}{2}))$, which ensures that $\|\nabla_u D^\eta_u f| \in L^2(B_{z_0}(\frac{R_N}{2}))$. By applying Theorem 1.5-(b) three times, we obtain that $|\nabla_x D^\eta_u f| \in L^2(B_{z_0}(\frac{R_N}{2^3}))$. This ends the proof of the induction $N+1$.

We iterate m times this induction and conclude that, for $r := \frac{r_0}{2^{3m}}$,

$$\sum_{\eta\in\mathbb{N}^d;|\eta|\leq m} \|D^\eta_x f\|_{L^2(B_{z_0}(r))} + \sum_{\kappa\in\mathbb{N}^d;|\kappa|\leq m+1} \|D^\kappa_u f\|_{L^2(B_{z_0}(r))} < +\infty.$$

Step 2. Finally, we estimate $\|\partial_t^2 f\|_{L^2(B_{z_0}(r))}$. Since $\nabla_x f$ and $g_{\frac{r_0}{2^3}}$ are in $L^2(B_{z_0}(\frac{r_0}{2^3}))$, according to Theorem 1.5-(a), we have

$$\begin{aligned}\|\partial_t f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq \|\partial_t f_{\frac{r_0}{2^3}} + (u\cdot\nabla_x f_{\frac{r_0}{2^3}})\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \|(u\cdot\nabla_x f_{\frac{r_0}{2^3}})\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}\\ &\leq C\|g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} < +\infty.\end{aligned}$$

Moreover $\partial_{u_i} f_{\frac{r_0}{2^3}}$ satisfies (3.8) and $\partial_{x_i} f_{\frac{r_0}{2^3}}$ satisfies

$$\partial_t\partial_{x_i} f_{\frac{r_0}{2^3}} - (u\cdot\nabla_x\partial_{x_i} f_{\frac{r_0}{2^3}}) - \frac{\sigma^2}{2}\triangle_u\partial_{x_i} f_{\frac{r_0}{2^3}} = \partial_{x_i} g_{\frac{r_0}{2^3}} \ \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))',$$

with $|\nabla_x f_{\frac{r_0}{2^3}}|$, $|\nabla_u f_{\frac{r_0}{2^3}}|$, $\partial^2_{x_i,x_j} f_{\frac{r_0}{2^3}}$, $\partial^2_{u_i,u_j} f_{\frac{r_0}{2^3}}$ and $\partial^2_{x_i,u_j} f_{\frac{r_0}{2^3}}$ in $L^2(B_{z_0}(\frac{r_0}{2^3}))$ for $1\leq i,j\leq d$. We easily deduce that $\partial_{x_i} g_{\frac{r_0}{2^3}}$ and $\partial_{u_i} g_{\frac{r_0}{2^3}}$ are also in $L^2(B_{z_0}(\frac{r_0}{2^3}))$. From Theorem 1.5-(a) again it follows that

$$\begin{aligned}\|\partial_t\partial_{x_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq C\|\partial_{x_i} g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x\partial_{x_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))},\\ \|\partial_t\partial_{u_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} &\leq C\|\partial_{u_i} g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x\partial_{u_i} f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))}.\end{aligned}$$

so that $|\partial_t\nabla_x f_{\frac{r_0}{2^3}}|$ and $|\partial_t\nabla_u f_{\frac{r_0}{2^3}}|$ are in $L^2(B_{z_0}(\frac{r_0}{2^3}))$. Now we observe that $\partial_t f_{\frac{r_0}{2^3}}$ satisfies

$$\partial_t^2 f_{\frac{r_0}{2^3}} - (u\cdot\nabla_x\partial_t f_{\frac{r_0}{2^3}}) - \frac{\sigma^2}{2}\triangle_u\partial_t f_{\frac{r_0}{2^3}} = \partial_t g_{\frac{r_0}{2^3}} \ \text{ in } (\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d))',$$

with

$$\partial_t g_{\frac{r_0}{2^3}} = \Gamma_{\frac{r_0}{2^3}}\partial_t f + (\Psi_{\frac{r_0}{2^3}}\cdot\nabla_u\partial_t f) + (\partial_t\Gamma_{\frac{r_0}{2^3}})f + \left(\partial_t\Psi_{\frac{r_0}{2^3}}\cdot\nabla_u f\right) \in L^2(B_{z_0}(\frac{r_0}{2^3})).$$

It follows that $\partial_t^2 f \in L^2(B_{z_0}(\frac{r_0}{2^4}))$ since

$$\|\partial_t^2 f_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} \leq C\|\partial_t g_{\frac{r_0}{2^3}}\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} + \frac{r_0}{2^3}\|\nabla_x(\partial_t f)\|_{L^2(B_{z_0}(\frac{r_0}{2^3}))} < +\infty.$$

This enables us to conclude on (3.5).

■

Lemma 3.3. *Let f be given as in Lemma 3.1. Then for a.e. $(t,x,u)\in Q_T$,*

$$\inf_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \wedge \inf_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u) \leq f(t,x,u) \leq \sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \vee \sup_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u).$$

Proof. Let $\{\eta_R\}_{R>0}$ be a sequence of $\mathcal{C}^\infty$-cutoff functions on $\mathbb{R}^d$ such that, for all $R>0$, $\eta_R = \eta_R(u)\in L^1\cap L^\infty(\mathbb{R}^d)$ and there exists $0<C_R<\infty$ such that

$$|\nabla_u\eta_R(u)| + |\triangle_u\eta_R(u)| \leq C_R\eta_R(u),\ \forall\, u\in\mathbb{R}^d,$$

(for instance, take $\eta_R(u) = R^2/(R^2+|u|^2)$). Taking $\lambda_\kappa(t) = \exp\{\kappa t\}$ for κ a real number that will be chosen later and

$$M = \sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \vee \sup_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u),$$

we get that

$$\begin{aligned}&L(\eta_R\lambda_\kappa\left|(f-M)^+\right|^2)\\ &=\left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) + \eta_R\lambda_\kappa L(\left|(f-M)^+\right|^2) - \sigma^2\lambda_\kappa\left(\nabla_u\eta_R\cdot\nabla_u\left|(f-M)^+\right|^2\right).\end{aligned}\tag{3.10}$$

Let us point out that the function $\triangle_u|(f-M)^+|^2$ is well defined a.e. on Q_T since, using Theorem 1.2, one can check that $\triangle_u|(f-M)^+|^2 = 2\nabla_u\cdot((f-M)^+\nabla_u(f-M)) = 2((f-M)^+\triangle_u(f-M)) + 2\left|\nabla_u(f-M)\right|^2 \mathbb{1}_{\{f>M\}}$. In particular

$$
\begin{aligned}
&L(\left|(f-M)^+\right|^2)\\
&= 2\left(\partial_t(f-M) - u\cdot\nabla_x(f-M) - b\cdot_u(f-M) - \frac{\sigma^2}{2}\triangle_u(f-M))\right)(f-M)^+ - \sigma^2\left|\nabla_u(f-M)\right|^2\\
&\leq 0,
\end{aligned}
$$

Therefore, integrating (3.10) over Q_T, we have

$$
\begin{aligned}
&\int_{Q_T} L(\eta_R\lambda_\kappa\left|(f-M)^+\right|^2)\\
&= \int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) + \eta_R\lambda_\kappa L(\left|(f-M)^+\right|^2) - \sigma^2\lambda_\kappa\left(\nabla_u\eta_R\cdot\nabla_u\left|(f-M)^+\right|^2\right)\\
&\leq \int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) - \sigma^2\lambda_\kappa\left(\nabla_u\eta_R\cdot\nabla_u\left|(f-M)^+\right|^2\right)
\end{aligned}
\tag{3.11}
$$

Observing that an integration by part on the second integral on the right-hand side of (3.11) gives

$$
\begin{aligned}
&\int_{Q_T} \left|(f-M)^+\right|^2 L(\eta_R\lambda_\kappa) - \sigma^2\lambda_\kappa\left(\nabla_u\eta_R\cdot\nabla_u\left|(f-M)^+\right|^2\right)\\
&= \int_{Q_T} \left(L(\eta_R\lambda_\kappa) + \sigma^2\lambda_\kappa\triangle_u\eta_R\right)\left|(f-M)^+\right|^2
\end{aligned}
$$

Using an integration by part for the left-hand side of (3.11) and, since

$$
|(f_0-M)^+| = 0 \text{ on } \mathcal{D}\times\mathbb{R}^d,\ \ |(q-M)^+| = 0 \text{ on } \Sigma_T^+.
$$

we get

$$
\begin{aligned}
\int_{Q_T} L(\eta_R\lambda_\kappa\left|(f-M)^+\right|^2) = &\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T)\left|(f(T)-M)^+\right|^2\\
&- \int_{\Sigma_T^-}(u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa\left|(\gamma(f)-M)^+\right|^2 + \int_{Q_T}(\nabla_u\cdot b)\,\eta_R\lambda_\kappa\left|(f-M)^+\right|^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T)\left|(f(T)-M)^+\right|^2 - \int_{\Sigma_T^-}(u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa\left|(\gamma(f)-M)^+\right|^2\\
&= \int_{Q_T}\left(L(\eta_R\lambda_\kappa) + \sigma^2\lambda_\kappa\triangle_u\eta_R - (\nabla_u\cdot b)\,\eta_R\lambda_\kappa\right)\left|(f-M)^+\right|^2\\
&= \int_{Q_T}\left(\kappa\eta_R - \nabla_u\eta_R\cdot b + \frac{\sigma^2}{2}\triangle_u\eta_R - (\nabla_u\cdot b)\,\eta_R\right)\lambda_\kappa\left|(f-M)^+\right|^2\\
&\leq \int_{Q_T}\left(\kappa + C_R(1+\frac{\sigma^2}{2} + \|b\|_{L^\infty}) + \|\nabla_u\cdot b\|_{L^\infty}\right)\eta_R\lambda_\kappa\left|(f-M)^+\right|^2.
\end{aligned}
$$

Since

$$
\int_{\mathcal{D}\times\mathbb{R}^d} \eta_R\lambda_\kappa(T)\left|(f(T)-M)^+\right|^2 - \int_{\Sigma_T^-}(u\cdot n_{\mathcal{D}}(x))\eta_R\lambda_\kappa\left|(\gamma(f)-M)^+\right|^2 \geq 0,
$$

choosing $\kappa < 0$ such that

$$\kappa + C_R(1 + \frac{\sigma^2}{2} + \|b\|_{L^\infty}) + \|\nabla_u \cdot b\|_{L^\infty} < 0$$

implies that $(f - M)^+ = 0$ and that $f \le M$ a.e. on Q_T. Replacing $f - M$ by $m - f$, for

$$m = \inf_{(x,u)\in\mathcal{D}\times\mathbb{R}^d} f_0 \wedge \inf_{(t,x,u)\in\Sigma_T-\Sigma_T^0} q(t,x,u),$$

and using similar arguments yields to $f \ge m$ a.e. on Q_T. ■

Proposition 3.4 (Continuity up to Σ_T^+). *Assume* (H_{PDE}) *and* ($H_{f_0,q}$). *Let* $f \in \mathcal{C}^{1,1,2}(Q_T)\cap\mathcal{C}([0,T];L^2(\mathcal{D}\times\mathbb{R}^d)) \cap \mathcal{H}(Q_T)$ *be the solution to* (6.5) *with inputs* (f_0, q). *Then* f *is continuous up to* Σ_T^+.

Proof. To show the continuity up to the boundary Σ_T^+, we follow the classical method of local barrier functions (see e.g. [Gilbarg and Trudinger, 2001]). Let $(t_0, x_0, u_0) \in \Sigma_T^+$ (i.e. $t_0 \in (0,T), x_0^{(d)} = 0, (u\cdot n_{\mathcal{D}}(x)) = -u_0^{(d)} > 0$). Since q is continuous in Σ_T^+, we can assume that for any $\epsilon > 0$, there exists a neighborhood $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$ such that

$$q(t_0,x_0,u_0) - \epsilon \le q(t,x,u) \le q(t_0,x_0,u_0) + \epsilon,\ \ \forall (t,x,u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap \Sigma_T^+.$$

In addition, since $u_0 \cdot n_{\mathcal{D}}(x_0) = -u_0^{(d)} > 0$, by reducing $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$, we can assume that $u \cdot n_{\mathcal{D}}(x) = -u^{(d)} > \eta > 0$ for all $(t,x,u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0}$. Consequently, by setting $\varrho(x)\colon x \in \mathbb{R}^d \mapsto \mathrm{dist}(x, \partial\mathcal{D})$ (which is simply $\varrho(x) = x^{(d)}$), and

$$L := \partial_t - (u\cdot\nabla_x) - (b(x,u)\cdot\nabla_u) - \frac{\sigma^2}{2}\triangle_u,$$

we observe that, for all $(t,x,u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0}$,

$$L(\varrho)(t,x,u) = -(u\cdot\nabla\varrho(x)) = u\cdot n_{\mathcal{D}}(x) > \eta > 0. \tag{3.12}$$

Reducing again $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$, we can assume that $\mathcal{O}^\epsilon_{t_0,x_0,u_0}$ has the form $(t_0-\delta_\epsilon, t_0+\delta_\epsilon)\times B_{x_0}(\delta'_\epsilon)\times B_{u_0}(\delta'_\epsilon)$ (where $B_{x_0}(\delta')$ [resp. $B_{u_0}(\delta')$] is the ball centered in x_0 [resp. u_0] of radius δ') for some positive constants $\delta_\epsilon, \delta'_\epsilon > 0$ chosen such that $0 \le t_0 - \delta_\epsilon < t_0 + \delta_\epsilon \le T$ and $\delta'_\epsilon < \eta$.

We can construct a maximizing barrier function related to $(t_0,x_0,u_0) \in \Sigma_T^+$ with

$$\overline{\omega}_\epsilon(x) = q(t_0,x_0,u_0) + \epsilon + k_\epsilon|x-x_0|^2 + K_\epsilon\varrho(x), \tag{3.13}$$

where the parameters $K_\epsilon, k_\epsilon > 0$ are chosen large enough so that, for M_ϵ^+ the upper-bound of f on $\partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$ (which is finite by Lemma 3.3), we have

$$\overline{\omega}_\epsilon(x) - M_\epsilon^+ \ge k_\epsilon|x-x_0|^2 - M_\epsilon^+ \ge k_\epsilon(\delta'_\epsilon)^2 - M_\epsilon^+ \ge 0,$$

and, by (3.12),

$$\begin{aligned} L(\overline{\omega}_\epsilon)(x) = -2k_\epsilon u\cdot(x-x_0) - K_\epsilon u\cdot\nabla\varrho(x) &\ge -2k_\epsilon|u||x-x_0| - K_\epsilon u^{(d)} \\ &\ge -2k_\epsilon(|u_0|+\delta')\delta' + K_\epsilon\eta \ge 0. \end{aligned}$$

In the same way, we construct a minimizing barrier of the form

$$\underline{\omega}_\epsilon(t,x,u) = q(t_0,x_0,u_0) - \epsilon - \tilde{k}_\epsilon|x-x_0|^2 - \tilde{K}_\epsilon\varrho(x). \tag{3.14}$$

with $\tilde{K}_\epsilon, \tilde{k}_\epsilon > 0$ chosen so that, for M_ϵ^- the lower-bound of f on $\partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$, we have

$$\underline{\omega}_\epsilon(x) - M_\epsilon^- \leq 0 \text{ and } L(\underline{\omega}_\epsilon)(x) \leq 0.$$

Thus, $\overline{\omega}_\epsilon$ and $\underline{\omega}_\epsilon$ satisfy the properties

$$\textit{(P)-}\begin{cases} (a)\ \overline{\omega}_\epsilon(t,x,u) \geq q(t,x,u) \geq \underline{\omega}_\epsilon(t,x,u) \text{ for all } (t,x,u) \in \mathcal{O}_{t_0,x_0,u_0} \cap (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d, \\ (b)\ L(\overline{\omega}_\epsilon) \geq 0 \geq L(\underline{\omega}_\epsilon) \text{ for all } (t,x,u) \in \mathcal{O}_{t_0,x_0,u_0} \cap Q_T, \\ (c)\ \overline{\omega}_\epsilon(t,x,u) \geq M^+ \geq f(t,x,u), \text{ and } \underline{\omega}_\epsilon(t,x,u) \leq M^- \leq f(t,x,u), \text{ for all } (t,x,u) \in \partial\mathcal{O}_{t_0,x_0,u_0} \cap Q_T, \\ (d)\ \lim\limits_{\epsilon\to 0^+} \overline{\omega}_\epsilon(t_0,x_0,u_0) = \lim\limits_{\epsilon\to 0^+} \underline{\omega}_\epsilon(t_0,x_0,u_0) = q(t_0,x_0,u_0). \end{cases}$$

Now we shall prove that, for f the solution to (6.5), $\underline{\omega}_\epsilon \leq f \leq \overline{\omega}_\epsilon$ on $\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$. Owing to the property *(P)-*(d), this allows to conclude that $f(t,x,u)$ tends to $q(t_0,x_0,u_0)$ as (t,x,u) tends to (t_0,x_0,u_0), for all (t_0,x_0,u_0) of Σ_T^+.

For the local comparison between $\overline{\omega}_\epsilon$ and f, we proceed as in the proof of Lemma 3.3 and we consider the positive part $(f - \overline{\omega}_\epsilon)^+$ of $f - \overline{\omega}_\epsilon$. Let β be a real parameter that we will specify later. Recalling from the proof of Lemma 3.3 that the function $\triangle_u|(f - \overline{\omega}_\epsilon)^+|^2$ is well defined a.e. on Q_T with

$$\triangle_u|(f - \overline{\omega}_\epsilon)^+|^2 = 2((f - \overline{\omega}_\epsilon)^+ \triangle_u(f - \overline{\omega}_\epsilon)) + 2\,|\nabla_u(f - \overline{\omega}_\epsilon)|^2\, \mathbb{1}_{\{f > \overline{\omega}_\epsilon\}}.$$

we shall observe that, on $\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$,

$$L(\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2) = \beta\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2 + \exp\{\beta t\} L(\left|(f - \overline{\omega}_\epsilon)^+\right|^2).$$

The property $(P-)(b)$ ensures that

$$L(\left|(f - \overline{\omega}_\epsilon)^+\right|^2) = 2(f - \overline{\omega}_\epsilon)^+ L(f - \overline{\omega}_\epsilon) - \sigma^2\,|\nabla_u(f - \overline{\omega}_\epsilon)|^2\,\mathbb{1}_{\{f > \overline{\omega}_\epsilon\}} \leq -\sigma^2\,|\nabla_u(f - \overline{\omega}_\epsilon)|^2\,\mathbb{1}_{\{f > \overline{\omega}_\epsilon\}} \leq 0,$$

so that

$$L(\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2) \leq \beta\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2.$$

Integrating the two sides of the above inequality over $\mathcal{O}_{t_0,x_0,u_0} \cap Q_T$, we get

$$\int_{\mathcal{O}_{t_0,x_0,u_0} \cap Q_T} L(\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2) \leq \int_{\mathcal{O}_{t_0,x_0,u_0} \cap Q_T} \beta\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2.$$

wing to $(P)-(a)$ and $(P)-(c)$, $(f - \overline{\omega}_\epsilon)^+$ is zero on $\mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap \Sigma_T$ and $\partial\mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap Q_T$. An integration by parts of the left-hand side expression yields

$$\begin{aligned} &\int_{\mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap Q_T} L(\exp\{\beta t\}\left|(f - \overline{\omega}_\epsilon)^+\right|^2)(t,x,u) \\ &= \int_{\mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap Q_T} \exp(\beta t)\left|(f - \overline{\omega}_\epsilon)^+\right|^2(t,x,u)\nabla_u \cdot b(x,u), \end{aligned}$$

and

$$0 \leq \int_{\mathcal{O}_{t_0,x_0,u_0} \cap Q_T} (\beta - \nabla_u \cdot b(x,u))\exp\{\beta t\}|(f - \overline{\omega}_\epsilon)^+|^2.$$

Choosing $\beta < 0$ such that $\beta + \|\nabla_u \cdot b\|_{L^\infty(\mathcal{D}\times\mathbb{R}^d)} < 0$ ensures that, for a.e. $(t,x,u) \in \mathcal{O}^\epsilon_{t_0,x_0,u_0} \cap Q_T$, $f(t,x,u) \leq \overline{\omega}_\epsilon(t,x,u)$. Similar arguments entail that $\underline{\omega}_\epsilon \leq f$. ■

Feynman-Kac representation and continuity up to and along Σ_T^-. We prove the Feynman-Kac representation (6.6) by replicating the arguments of Friedman [Friedman, 2012, Chapter 5, Theorem 5.2]: for $(y,v)\in\mathcal{D}\times\mathbb{R}^d$ fixed, let $((x_t^{y,v},u_t^{y,v});\,t\in[0,T])$ satisfy (6.4). Set $\beta_\delta^{y,v}:=\inf\{t>0\,;\,d(x_t^{y,v},\partial\mathcal{D})\leq\delta\}$. Since f is smooth in the interior of Q_T and satisfies (6.5), applying Itô's formula to $f(t-s,x^{y,v}_{s\wedge\beta_\delta^{y,v}},u^{y,v}_{s\wedge\beta_\delta^{y,v}})$, for $s\in[0,t]$, yields

$$f(t,y,v)=\mathbb{E}_{\mathbb{P}}\left[f_0(x_t^{y,v},u_t^{y,v})\mathbb{1}_{\{t\,\leq\,\beta_\delta^{y,v}\}}\right]+\mathbb{E}_{\mathbb{P}}\left[f(t-\beta_\delta^{y,v},x^{y,v}_{\beta_\delta^{y,v}},u^{y,v}_{\beta_\delta^{y,v}})\mathbb{1}_{\{t\,>\,\beta_\delta^{y,v}\}}\right].$$

Since $\mathbb{P}$-a.s., $\beta_\delta^{y,v}$ tends to $\beta^{y,v}=\inf\{t>0\,;\,d(x_t^{y,v},\partial\mathcal{D})=0\}$, as δ tends to 0, and thanks to Proposition 3.4, one obtains (6.6).

Proposition 3.5. *Assume ($H_{f_0,q}$). Let $f\in\mathcal{C}^{1,1,2}(Q_T)\cap\mathcal{C}(Q_T\cup\Sigma_T^+)$ be the solution to (6.5). Then f is continuous along and up to Σ_T^-.*

Proof. According to (6.6) and since f_0 and q are continuous, the continuity of f up to Σ_T^- will follow from the continuity of $(y,v)\mapsto(\beta^{y,v},x_t^{y,v},u_t^{y,v})$. $\mathbb{P}$-almost surely, for all $t\geq0$, the flow $(y,v)\mapsto(x_t^{y,v},u_t^{y,v})$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$. As $(y,v)\notin\Sigma^0\cup\Sigma^+$, we have $\beta^{y,v}=\tau^{y,v}:=\inf\{t>0;\,x_t^{y,v}\notin\overline{\mathcal{D}}\}$. To prove that $(y,v)\mapsto\tau^{y,v}$ is continuous up to Σ^-, we follow the general proof of the continuity of exit time related to a flow of continuous processes given in Proposition 6.3 in Darling and Pardoux [Darling and Pardoux, 1997]. First, replicating the argument of the authors, one can show that, for all $(y_m,v_m)\in\mathcal{D}\times\mathbb{R}^d$ such that $\lim_{m\to+\infty}(y_m,v_m)=(y,v)\in\Sigma^-$,

$$\limsup_{m\to+\infty}\tau^{y_m,v_m}\leq\tau^{y,v}.$$

Next, it is sufficient to check that

$$\tau^{y,v}\leq\liminf_{m\to+\infty}\tau^{y_m,v_m}.$$

By an [Bossy and Jabir, 2015] it is shown that for a.e. $(y,v)\in\mathcal{D}\times\mathbb{R}^d\cup\Sigma^-$, the path $t\mapsto(x_t^{y,v},u_t^{y,v})$ never hits $\Sigma^0\cup\Sigma^-$, and, since $\mathbb{P}$-a.s. $(t,y,v)\mapsto(x_t^{y,v},u_t^{y,v})$ is continuous on $[0,+\infty)\times\overline{\mathcal{D}}\times\mathbb{R}^d$, one can check that

$$\overline{\{(x^{y_m,v_m}_{\tau^{y_m,v_m}},u^{y_m,v_m}_{\tau^{y_m,v_m}});\,m\in\mathbb{N}\}}\subset\Sigma^+,$$

and that $(x^{y,v}_{\liminf_{m\to+\infty}\tau^{y_m,v_m}},u^{y,v}_{\liminf_{m\to+\infty}\tau^{y_m,v_m}})\in\Sigma^+$. Since $\tau^{y,v}=\inf\{t>0;\,(x_t^{y,v},u_t^{y,v})\in\Sigma^+\}$, we deduce that $\tau^{y,v}\in[0,\liminf_{m\to+\infty}\tau^{y_m,v_m}]$. ∎

3.2 Proof of Corollary 6.2

Proof. For $n>1$, let us assume that $\Gamma^\psi_{n-1}\in\mathcal{C}(\overline{Q_T}\setminus\Sigma^0)$ with $\Gamma^\psi_{n-1}|_{\Sigma_T^-}\in L^2(\Sigma_T^-)$. Then $\Gamma^\psi_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))|_{\Sigma_T^+}$ is in $L^2(\Sigma_T^+)$ since, by using the change of variables

$$u\mapsto\widehat{u}:=u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)$$

for fixed $x\in\partial\mathcal{D}$, we have

$$\begin{aligned}&\int_{\Sigma_T^+}|(u\cdot n_{\mathcal{D}}(x))|\left(\Gamma^\psi_{n-1}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))\right)^2d\lambda_{\Sigma_T}(t,x,u)\\&=\int_{\Sigma_T^-}|(u\cdot n_{\mathcal{D}}(x))|\left(\Gamma^\psi_{n-1}(t,x,u)\right)^2d\lambda_{\Sigma_T}(t,x,u)=\|\Gamma^\psi_{n-1}\|^2_{L^2(\Sigma_T^-)}<+\infty.\end{aligned}\tag{3.15}$$

From the strong Markov property of the solution of (1.1), we get that for all $(x,u)\in(\mathcal{D}\times\mathbb{R}^d)\cup\Sigma^+$,

$$\mathbb{E}[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}},U^{x,u}_{t\wedge\tau_n^{x,u}})\mathbb{1}_{\{\tau_1^{x,u}<t\}}]=\mathbb{E}[\Gamma^\psi_{n-1}(t-\tau_1^{x,u},X^{x,u}_{\tau_1^{x,u}},U^{x,u}_{\tau_1^{x,u}})\mathbb{1}_{\{\tau_1^{x,u}<t\}}].\tag{3.16}$$

Considering a sequence $(x_m, u_m, m \in \mathbb{N})$ in $\mathcal{D} \times \mathbb{R}^d$ converging to $(x,u) \in \Sigma^+$, and $t > 0$, we have, for m large enough

$$(\tau_1^{x_m,u_m}, X_t^{x_m,u_m}, U_t^{x_m,u_m}, \{t < \tau_1^{x_m,u_m}\}) = (\beta^{x_m,u_m}, x_t^{x_m,u_m}, u_t^{x_m,u_m}, \{t < \beta^{x_m,u_m}\})$$
$$(X^{x_m,u_m}_{\tau_1^{x_m,u_m}}, U^{x_m,u_m}_{\tau_1^{x_m,u_m}}) = (x^{x_m,u_m}_{\beta^{x_m,u_m}}, \widehat{u}^{x_m,u_m}_{\beta^{x_m,u_m}}).$$

Hence, from the continuity of $(y,v) \mapsto (\beta^{y,v}, x_t^{y,v}, u_t^{y,v})$ proved with Proposition 3.5,

$$\lim_{m\to+\infty} (\tau_1^{x_m,u_m}, X^{x_m,u_m}_{t\wedge\tau_1^{x_m,u_m}}, U^{x_m,u_m}_{t\wedge\tau_1^{x_m,u_m}}) = (0, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)).$$

Since Ψ is continuous and $\Psi = 0$ on Σ^+, the right-hand side of (3.16) is then continuous on $(\mathcal{D} \times \mathbb{R}^d) \cup \Sigma^+$, as well as

$$\mathbb{E}[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\mathbb{1}_{\{\tau_1^{x,u}\geq t\}}] = \mathbb{E}[\psi(X^{x,u}_{t\wedge\tau_1^{x,u}}, U^{x,u}_{t\wedge\tau_1^{x,u}})\mathbb{1}_{\{\tau_1^{x,u}\geq t\}}].$$

Moreover, for $(t,x,u) \in \Sigma_T^+$,

$$\begin{aligned}\Gamma_n^\psi(t,x,u) &= \lim_{m\to+\infty} \left\{\mathbb{E}[\psi(X^{x_m,u_m}_{t\wedge\tau_n^{x_m,u_m}}, U^{x_m,u_m}_{t\wedge\tau_n^{x_m,u_m}})\mathbb{1}_{\{\tau_1^{x_m,u_m}<t\}}] + \mathbb{E}[\psi(X^{x_m,u_m}_{t\wedge\tau_n^{x_m,u_m}}, U^{x_m,u_m}_{t\wedge\tau_n^{x_m,u_m}})\mathbb{1}_{\{\tau_1^{x_m,u_m}\geq t\}}]\right\} \\ &= \Gamma_{n-1}^\psi(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)).\end{aligned}$$

Now Theorem 6.1 ensures that there exists a classical solution f_n to (6.5) for $f_0 = \psi$ and $q(t,x,u) = \Gamma_{n-1}^\psi(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ on Σ_T^+. According to (6.6), we have, for $(t,x,u) \in Q_T$

$$\begin{aligned}f_n(t,x,u) &= \mathbb{E}_{\mathbb{P}}\left[\psi(x_t^{x,u}, u_t^{x,u})\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}\left[\Gamma_{n-1}^\psi\left(t-\beta^{x,u}, x^{x,u}_{\beta^{x,u}}, u^{x,u}_{\beta^{x,u}} - 2(u^{x,u}_{\beta^{x,u}}\cdot n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}}))n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}})\right)\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right].\end{aligned}$$

One can observe that

$$\mathbb{E}_{\mathbb{P}}\left[\psi(x_t^{x,u}, u_t^{x,u})\mathbb{1}_{\{t\,\leq\,\beta^{x,u}\}}\right] = \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u}, U_t^{x,u})\mathbb{1}_{\{t\,\leq\,\tau_1^{x,u}\}}\right] = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\mathbb{1}_{\{t\,\leq\,\tau_1^{x,u}\}}\right]$$

and that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\Gamma_{n-1}^\psi\left(t-\beta^{x,u}, x^{x,u}_{\beta^{x,u}}, u^{x,u}_{\beta^{x,u}} - 2(u^{x,u}_{\beta^{x,u}}\cdot n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}}))n_{\mathcal{D}}(x^{x,u}_{\beta^{x,u}})\right)\mathbb{1}_{\{t\,>\,\beta^{x,u}\}}\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\Gamma_{n-1}^\psi(t-\tau_1^{x,u}, X^{x,u}_{\tau_1^{x,u}}, U^{x,u}_{\tau_1^{x,u}})\mathbb{1}_{\{t\,>\,\tau_1^{x,u}\}}\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\mathbb{1}_{\{t\,>\,\tau_1^{x,u}\}}\right],\end{aligned}$$

where the second equality follows from the strong Markov property of $(X_t^{x,u}, U_t^{x,u})$. Therefore

$$f_n(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\right] = \Gamma_n^\psi(t,x,u),$$

from which we deduce that $\Gamma_n^\psi \in \mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T} \setminus \Sigma_T^0) \cap L^2((0,T)\times\mathcal{D}; H^1(\mathbb{R}^d))$ is a solution to (6.8) with $\Gamma_n^\psi|_{\Sigma_T^-} \in L^2(\Sigma_T^-)$. Moreover, according to (3.15), for all $t \in (0,T)$,

$$\begin{aligned}&\|\Gamma_n^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_n^\psi\|^2_{L^2(\Sigma_t^-)} + \int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma_n^\psi)^2 + \sigma^2\|\nabla_u\Gamma_n^\psi\|^2_{L^2(Q_t)} \\ &= \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma_t^+)} = \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_{n-1}^\psi\|^2_{L^2(\Sigma_t^-)}.\end{aligned} \tag{3.17}$$

which implies

$$\|\Gamma^\psi_n(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma^\psi_n\|^2_{L^2(\Sigma^-_t)} + \sigma^2\|\nabla_u\Gamma^\psi_n\|^2_{L^2(Q_t)}$$
$$\leq \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma^\psi_{n-1}\|^2_{L^2(\Sigma^-_t)} + \|\nabla_u\cdot b\|_{L_\infty(\mathcal{D}\times\mathbb{R}^d)}\int_{Q_t}(\Gamma^\psi_n)^2.$$

resulting in (6.9) by Gronwall's lemma. For $n=1$, setting $f_0=\psi$ and $q=\psi|_{\Sigma^+_T}=0$ (since ψ has its support in the interior of $\mathcal{D}\times\mathbb{R}^d$), one can check that $\Gamma^\psi_1\in\mathcal{C}^{1,1,2}_b(Q_T)\cap\mathcal{C}(\overline{Q_T}\setminus\Sigma^0_T)\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ satisfies (6.8) and (6.9). By induction, we end the proof. ■

3.3 Proof of Corollary 6.3

Proof. We first observe that since $\psi|_{\partial\mathcal{D}\times\mathbb{R}^d}=0$,

$$\Gamma^\psi_n(t,x,u)=\mathbb{E}_\mathbb{P}\left[\psi(X^{x,u}_{t\wedge\tau^{x,u}_n},U^{x,u}_{t\wedge\tau^{x,u}_n})\right]=\mathbb{E}_\mathbb{P}\left[\psi(X^{x,u}_t,U^{x,u}_t)\mathbb{1}_{\{\tau^{x,u}_n\geq t\}}\right].$$

Next, there exists a nonnegative function $\beta\in L^2(\mathbb{R}\times\mathbb{R})$ such that $\beta(|x|,|u|)=1$ on the support of ψ and $|\psi|\leq C\beta(|x|,|u|)$, with $C:=\sup_{(x,u)\in\mathcal{D}\times\mathbb{R}^d}|\psi|(x,u)$. Then

$$\Gamma^\psi_n(t,x,u)\leq C\mathbb{E}_\mathbb{P}\left[\beta(|X^{x,u}_t|,|U^{x,u}_t|)\mathbb{1}_{\{\tau^{x,u}_n\geq t\}}\right].$$

As $\mathbb{E}_\mathbb{P}[\beta(|U^{x,u}_t|)]$ is equal to the convolution product $(G*\beta)(|x|,|u|)$, where G denotes the density of the free Langevin process (6.4), we obtain

$$-C(G*\beta)(|x|,|u|)\leq\Gamma^\psi_n(t,x,u)\leq C(G*\beta)(|x|,|u|),\ \text{on } Q_T. \tag{3.18}$$

Owing to the continuity of Γ^ψ_n, from the interior of Q_T to its boundary, (3.18) still holds true along $\Sigma^\pm_T$.

It is show in **Proposition 3.1** from [Bossy and Jabir, 2011] that for a.e. $(x,u)\in(\mathcal{D}\times\mathbb{R}^d)\cup(\Sigma\setminus\Sigma^0)$, $\mathbb{P}_{(x,u)}$-a.s. τ_n grows to ∞ as n increases, so then

$$\lim_{n\to+\infty}\Gamma^\psi_n(t,x,u)=\Gamma^\psi(t,x,u),\ \text{for a.e. } (t,x,u)\in Q_T,\ \lambda_{\Sigma_T}\text{-a.e. } (t,x,u)\in\Sigma_T\setminus\Sigma^0_T. \tag{3.19}$$

Indeed,

$$\left|\Gamma^\psi_n(t,x,u)-\Gamma^\psi(t,x,u)\right|=\left|\mathbb{E}_\mathbb{P}\left[\psi(X^{x,u}_t,U^{x,u}_t)\mathbb{1}_{\{\tau^{x,u}_n\leq t\}}\right]\right|\leq\|\psi\|_\infty\mathbb{P}(\tau^{x,u}_n\leq t).$$

In particular, (3.18) is also true for $\Gamma^\psi(t)$. We conclude by the Lebesgue Dominated Convergence Theorem that $\Gamma^\psi_n(t)$ converges to $\Gamma^\psi(t)$ in $L^2(\mathcal{D}\times\mathbb{R}^d)$. And since Γ^ψ_n is continuous on the compact $[0,T]$ we have the convergence to Γ^ψ in $L^2(Q_T)$. The Lebesgue Dominated Convergence Theorem also shows that:

$$\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma^\psi_n)^2\to\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma^\psi)^2.$$

Next we deduce that the norms involving Γ^ψ_n in the left-hand side of (3.17) are finite for all t, uniformly in n (as the right-hand side of (6.9) is bounded uniformly in n by the Maxwellian bound (3.18) and ψ is of compact support). Therefore, the estimate (3.17) is also true for Γ^ψ (see e.g. [Brezis, 2010]), and Γ^ψ_n converges to Γ^ψ in $L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ and the equality (3.17) becomes:

$$\|\Gamma^\psi(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)}+\int_{Q_t}(\nabla_u\cdot b(x,u))(\Gamma^\psi)^2+\sigma^2\|\nabla_u\Gamma^\psi\|^2_{L^2(Q_t)}=\|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} \tag{3.20}$$

and by Gronwall's lemma, we obtain (6.10). ■

Chapter 2

Empirical Analysis Based on Numerical Experiments

1 Introduction

A quick review of main existing theoretical convergence results

The time discretisation of diffusion processes introduces errors which depend on the type of scheme considered, the regularity of the coefficients of the stochastic differential equation, the type of boundary conditions, the type of error considered (e.g.: weak, strong) and other factors.

Let us consider a general d dimensional SDE:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \qquad X_0 = x_0$$

x_0 an L^p random variable and a time discretisation on the uniform grid $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$. If b and σ verify that there exist $\beta \in (0, 1]$ and a constant $C_{b,\sigma,T}$ such that for any $x, y \in \mathbb{R}^d$, $s, t \in \mathbb{R}^+$ we have that:

$$|b(t,x) - b(s,y)| + \|\sigma(t,x) - \sigma(y,v)\| \leq C_{b,\sigma,T} \left(|t-s|^\beta + |x-y|\right)$$

then:

$$\left\| \sup_{k\in\{0,\ldots,N\}} \left|X_{t_k} - \bar{X}_{t_k}\right| \right\|_{L^p(\Omega)} \leq K_{p,b,\sigma,T} \left(1 + \|X_0\|_{L^p(\Omega)}\right) \left(\frac{T}{N}\right)^{\beta\wedge\frac{1}{2}},$$

where $(\bar{X})_{t_0,\ldots,t_N}$ is the Euler-Maruyama discretisation of $(X_t)_{0\leq t\leq T}$. If σ is a constant and $\beta = 1$, then the above strong error is of order $\mathcal{O}\left(\frac{T}{N}\right)$, result which is proven using a Milstein scheme which coincides in this case with the Euler-Maruyama scheme as in [Pagès, 2017].

In the case of the weak error, the bounds that are obtained depend also on the regularity of the test function.

[Talay and Tubaro, 1989] showed that, if b and σ are infinitely differentiable with bounded derivatives and f is also an infinitely differentiable function having at most polynomial growth, then for every positive integer $R \in \mathbb{N}^*$:

$$\mathcal{E}_{R+1} \equiv \mathbb{E}f(X_T^x) - \mathbb{E}f(\bar{X}_T^{x,N}) = \sum_{k=1}^{R} \frac{c_k}{N^k} + \mathcal{O}\left(N^{-(R+1)}\right), \quad \text{as } N \to \infty$$

where the real valued coefficients c_k depend on f, T, b and σ and again $(\bar{X}_{t_k})_{k=\{0,\ldots,N\}}$ is obtained through the Euler-Maruyama scheme.

If σ is uniformly elliptic, then by [Bally and Talay, 1996], we have that the same expansion as previously presented applies for any f that is just measurable bounded. These results are obtained by utilising the regularity of the semigroup associated with the SDE and Richardson-Romberg extrapolations.

If we introduce boundary conditions, then once more we can have different convergence rates.

[Gobet, 2000] analysed the case with absorbing boundary conditions. In the case of a domain with C^3 boundaries, C^3 SDE coefficients and uniformly elliptic diffusion coefficient, we have for every bounded measurable test function f, vanishing at the boundary, that:

$$\mathbb{E}\left(f(X_T)\mathbb{1}_{\tau(X)>T}\right) - \mathbb{E}\left(f(\bar{X}_T^N)\mathbb{1}_{\tau(\bar{X}^N)>T}\right) = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

where τ is the exit time of the domain.

If we impose further regularity on the coefficients and we consider the time continuous Euler scheme, then above difference becomes of order $\mathcal{O}\left(\frac{1}{N}\right)$. These results utilise estimations of the transition density of the killed diffusion, as ones from [Ladyzhenskaia and Ural'ceva, 1968]. A further extension was given by [Gobet and Menozzi, 2004] in the case of hypo-elliptic diffusions but with uniform ellipticity at the absorbing boundary.

In the case of uniformly elliptic stochastic differential equations with reflecting boundary conditions, we have the following generic equation:

$$\left\{ x_t = x_0 + \int_0^t b(x_s)\, ds + \int_0^t \sigma(x_s)\, dW_s + L_t \right.,$$

where L represents the local time. [Lépingle, 1995] bounded the strong error of a continuous version of the Euler-Maruyama scheme by the square-root of the time step by using the distribution of the drifted Brownian and its maximum on each time step. [Bossy *et al.*, 2004] introduced a scheme that also applies for oblique reflections and has the weak error bounded linearly in the time step.

In a different setting, penalised schemes that converge towards reflected processes have been proposed. The general principle is the process is allowed go over the boundaries of its domain, but each time that happens, it incurs a penalisation that forces it to return to the original domain and in the limit the penalisation becomes stronger. In the case of deterministic kinetic models, [Paoli and Schatzman, 1993] proposed a penalised scheme solely on the velocity component and showed that it converges towards a specularly reflected process when the penalisation goes to infinity. In a probabilistic setting, [Slominski, 2013] offered a different penalisation scheme, on the whole process and a strong convergence rate.

Returning to our case

We recall that we are considering the process $(X_t, U_t)_{0\le t\le T} \in \mathcal{D}\times\mathbb{R}^d$ defined as the solution of the equation:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s\, ds, \\ U_t = U_0 + \displaystyle\int_0^t b(X_s, U_s)\, ds + \sigma W_t + K_t, \\ K_t = -\displaystyle\sum_{0<s\le t} 2\left(U_{s^-}\cdot n_{\mathcal{D}}(X_s)\right) n_{\mathcal{D}}(X_s)\mathbb{1}_{\{X_s\in\partial\mathcal{D}\}}, \end{cases} \tag{1.1}$$

where $\mathcal{D} := \mathbb{R}^{d-1}\times(0,+\infty)$, and the drift b verifies the hypotheses (*H*) and and we also consider the time discretisation presented in the previous chapter.

In this chapter we analyse from a numerical point of view, the theoretical convergence rate of the weak error presented in Theorem 1.6 on a panel of test cases:

- where there exists an explicit solution

- where the hypotheses for the test function and the drift are satisfied
- where the hypothesis on the drift (*H*) is not verified

We also consider several penalised schemes that exist in the literature. And to extend the results, a numerical analysis of the Richardson-Romberg extrapolation is given for the test cases that are considered to see if an expansion of the weak error would be possible. In addition, the numerical strong error convergence rates are estimated and used afterwards in the simulation of a multi-level Monte Carlo procedure. The results of this multi level procedure are compared to the simple simulation of the symmetrised scheme.

2 Reflection: symmetrised scheme

Let $0 < t_0 < t_1 < \ldots < t_n = T$ be a uniform mesh of the interval $[0, T]$ and consider the discretisation scheme of (1.1), already presented in (1.4)-(1.6), which will be called the symmetrised scheme:

$$\textbf{Discretisation of } \boldsymbol{X} \begin{cases} \bar{Y}_{t_{i+1}} = \bar{X}^c_{t_i} + (t_{i+1} - t_i)\bar{U}^c_{t_i} \\ \bar{X}^c_{t_{i+1}} = |\bar{Y}_{t_{i+1}}| \end{cases} \tag{2.1}$$

$$\textbf{Discretisation of } \boldsymbol{U} \begin{cases} \textbf{if } \exists \theta_i = t_i - \dfrac{\bar{X}^c_{t_i}}{\bar{U}^c_{t_i}} \in (t_i, t_{i+1}): \\ \quad \text{for } t_i \le t < \theta_i \\ \qquad \bar{U}^c_t = \bar{U}^c_{t_i} + b(\bar{X}^c_{t_i}, \bar{U}^c_{t_i})(t - t_i) + \sigma(W_t - W_{t_i}) \\ \quad \text{reflection}: \\ \qquad \bar{U}^c_{\theta_i} = -\bar{U}^c_{\theta_i^-} \\ \quad \text{for } \theta_i \le t < t_{i+1}: \\ \qquad \bar{U}^c_t = \bar{U}^c_{\theta_i} + b(\bar{X}^c_{\theta_i}, \bar{U}^c_{\theta_i})(t - \theta_i) + \sigma(W_t - W_{\theta_i}) \\ \textbf{else}: \\ \quad \text{for } t_i \le t < t_{i+1}: \\ \qquad \bar{U}^c_t = \bar{U}^c_{t_i} + b(\bar{X}^c_{t_i}, \bar{U}^c_{t_i})(t - t_i) + \sigma(W_t - W_{t_i})\,. \end{cases} \tag{2.2}$$

We recall that a collision takes place during a time interval, if there exists $\theta_i \in (t_i, t_{i+1})$ such that $\theta_i = t_i - \dfrac{\bar{X}^c_{t_i}}{\bar{U}^c_{t_i}}$. According to this definition, then $\bar{Y}_{\theta_i} = \bar{X}^c_{\theta_i} = 0$. In the simulation procedure, we will just consider this scheme at the discretisation times $(t_i)_{i=\{0,\ldots,n\}}$ and collision times $(\theta_i)_{i=\{0,\ldots,n\}}$ which are $\mathcal{F}_{t_i}-$measurable. We mention once again that this scheme supposes that only one collision takes place per time-step.

Weak Error

We recall that weak error is defined for any $T > 0$, as $\left|\mathbb{E}f(X^c_T, U^c_T) - \mathbb{E}f(\bar{X}^c_T, \bar{U}^c_T)\right|$ where f is a smooth enough function that verifies the specularity condition, the process $(X^c_t, U^c_t)_{0\le t\le T}$ verifies the SDE (1.1) and $(\bar{X}^c_t, \bar{U}^c_t)_{0\le t\le T}$ is its time-dicretisation, with time step Δt. We denote $\text{Error}[f](\Delta t) := \left|\mathbb{E}f(X^c_T, U^c_T) - \mathbb{E}f(\bar{X}^c_T, \bar{U}^c_T)\right|$. For anything but the most trivial cases, one cannot compute the two expectations so Monte Carlo simulations are required in order to estimate this quantity. Instead, we consider the following error $\overline{\text{Error}}[f](\Delta t)$, which we decompose as the bias and variance:

$$
\begin{aligned}
\overline{\text{Error}}[f](\Delta t)^2 &:= \mathbb{E}\left(\mathbb{E}f(X_T^c, U_T^c) - \frac{1}{N_{\text{MC}}}\sum_{n=1}^{N_{\text{MC}}} f(\bar{X}_T^{c,n}, \bar{U}_T^{c,n})\right)^2 \\
&= (\text{Error}[f](\Delta t))^2 + \mathbb{E}\left(\mathbb{E}f(\bar{X}_T^c, \bar{U}_T^c) - \frac{1}{N_{\text{MC}}}\sum_{n=1}^{N_{\text{MC}}} f(\bar{X}_T^{c,n}, \bar{U}_T^{c,n})\right)^2 ,
\end{aligned} \tag{2.3}
$$

where $(\bar{X}_T^{c,n}, \bar{U}_T^{c,n})_{n=1,\dots,N_{\text{MC}}}$ are N_{MC} independent copies of $(\bar{X}_T^c, \bar{U}_T^c)$. The second term represents the statistical error produced when replacing $\mathbb{E}f(\bar{X}_T^c, \bar{U}_T^c)$ by its Monte Carlo estimator. This error needs to be reduced sufficiently in order for the weak error to dominate and be observed in our experiments. One final approximation is calculating

$$
\overline{\overline{\text{Error}}}[f](\Delta t)^2 := \frac{1}{N_{\overline{\overline{\text{Err}}}}}\sum_{i=0}^{N_{\overline{\overline{\text{Err}}}}}\left(\mathbb{E}f(X_T^c, U_T^c) - \frac{1}{N_{\text{MC}}}\sum_{n=1}^{N_{\text{MC}}} f(\bar{X}_T^{c,n,i}, \bar{U}_T^{c,n,i})\right)^2 . \tag{2.4}
$$

The independent copies $(\bar{X}_T^{c,n,i}, \bar{U}_T^{c,n,i})_{n=1,\dots,N_{\text{MC}}, i=1,\dots,N_{\overline{\overline{\text{Err}}}}}$ are obtained with a time discretisation value Δt and we plot $\Delta t \mapsto \overline{\overline{\text{Error}[f]}}(\Delta t)$.

2.1 Implementation

The simulation procedure (which is similar throughout this chapter) is the following:

Data:

- T : Final time
- $n_{\Delta t}$: number of different time-step sizes
- N_{MC}: the number of MC estimator trajectories
- $N_{\overline{\overline{\text{Err}}}}$: the number of error estimators
- f : test function

Result: List of $\overline{\overline{\text{Error}}}[f](\Delta t)$ for $\Delta t = T \cdot 2^{-i}$ for $i = \{0, \dots, n_{\Delta t}\}$
for *all independent trajectories needed i.e.* $N_{MC} \times N_{\overline{\overline{Err}}}$ **do**

- Simulate the finest Brownian trajectory with time-step $T \cdot 2^{-n_{\Delta t}}$: $(W_{t_0}, W_{t_1}, W_{t_2} \dots, W_{t_{N-2}}, W_{t_{N-1}}, W_{t_N})$ and store as list L^W;
- Use the brownian increments in L^W and formulas (2.1)-(2.2) to calculate $(\bar{X}_T^{c,T\cdot 2^{-n_{\Delta t}}}, \bar{U}_T^{c,T\cdot 2^{-n_{\Delta t}}})$;
- If a collision takes places at $\theta_k \in]t_k, t_{k+1}[$, simulate W_{θ_k} through a Brownian bridge and save in a list L^θ, needed for (2.2);

  for $j = n_{\Delta t} - 1; j \geq 0; j = j - 1$ **do**

  - Take the Brownian trajectory with time-step $T \cdot 2^{-j}$ obtained by considering only the 2^j-th terms in the list L^W;
  - Iterate over this trajectory using formulas (2.1)-(2.2) to calculate $(\bar{X}_T^{c,T\cdot 2^{-j}}, \bar{U}_T^{c,T\cdot 2^{-j}})$;
  - If a collision takes place, a Brownian bridge is simulated using the starting and ending values from L^W and L^θ, which is needed for (2.2);

  end

end

- Calculate $\overline{\overline{\text{Error}}}[f](\Delta t)$ for $\Delta t = T \cdot 2^{-i}$ for $i = \{0, \dots, n_{\Delta t}\}$;

Algorithm 1: Simulation algorithm

It can be seen that there is a loop on trajectories and afterwards a loop on the list of different time-step sizes. This was done in order to simplify the parallelisation which was performed by splitting the trajectories on each processor. The workload is fairly balanced between independent trajectories. If the other obvious route had been taken, namely having the loop on the list of time-steps before the loop on the trajectories then either (a) the parallelisation would be performed on the time-step loop meaning that the workload would be unbalanced as one processor would need to simulate trajectories with one value of Δt while another processor would get a different value of Δt thus having the one with the smallest time-step needing to perform more computations *or* (b) the parallelisation would be performed on the trajectory loop, which would increase the number of communications as the data would be aggregated each time a new Δt is considered.

An added benefit for choosing this form of algorithm is that the trajectories for different time-step sizes are correlated and therefore random number generator is not used very often but most importantly, the Richardson-Romberg estimators are also correlated meaning that the variance does not increase as seen in [Pagès, 2017].

The reason for simulating from the finest to the coarsest discretised trajectory, instead of the other way round, is because the collision times need to be computed for each trajectory. In a coarse to fine approach Brownian bridges need to be simulated, and these bridges would need to contain all the collision times from all the coarser trajectories. In a fine to coarse approach, this is avoided as only the collision times of the finest trajectory need to be stored.

2.2 Test cases description

For the test functions $f\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto x^2 + u^2$ and also $g\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \exp\left(-(x-0.5)^2 - u^2\right)$. Concerning the drift, we consider 3 different examples:

- Brownian case: $b\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto 0$
- sin case: $b\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto -\sin(2\pi x) + \dfrac{1}{2}\sin(2\pi u)$
- Ornstein-Uhlenbeck case: $b\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto -5(u+5)$

In the case of no drift, due to the choice of f, it is possible to calculate $\mathbb{E} f(X_T^c, U_T^c) = \mathbb{E} f(X_T^f, U_T^f)$ analytically. We have that $\mathbb{E}\left[(X_T^{x,u})^2 + (U_T^{x,u})^2\right] = \dfrac{T^3}{3} + T^2u^2 + 2Tux + u^2 + x^2 + T$.

Initial points $x = 0.1, u = -1.1$, $x = 0.01, u = -0.11$ and finally $x = 0.001, u = -0.011$. The initial values for the velocity are chosen negative and much larger than the initial values of the position in order to increase the probability that the process touches the specular boundary. The results are considered for $N_{\text{MC}} = 10^8$ trajectories and $N_{\text{Err}} = 10$ and for $\Delta t = 2^{-12}, \ldots, 2^{-5}$ are presented in log-log plots.

2.3 Outputs

Brownian case for the velocity

In the first case, where the drift $b \equiv 0$, we have the log-log plots in (2.1) and more precisely in table 14.

> **How to read the plot graphs**
> We explain how to read the log-log plots that appear in this section and in the next. The same rules apply to all the test cases.
>
> On the $x-$axis we have Δt, which decreases from right to left and on the $y-$axis we have the

values of the error. The dashed line that splits the plots from line to the upper right corner to the lower left corner represents the identity function. Since both axes are logarithmic, this line is a visual test to compare the slopes of other monomials.

The red dots and the light grey crosses represent the calculated error points while the error bars represent the 95% confidence interval. To estimate an order of convergence, we only consider the red points. We also print tables with the data that is plotted. The values in grey are exactly the values that are excluded from the estimations.

The dark line represents a simple linear regression between the red points. We indicate its slope in the legend.

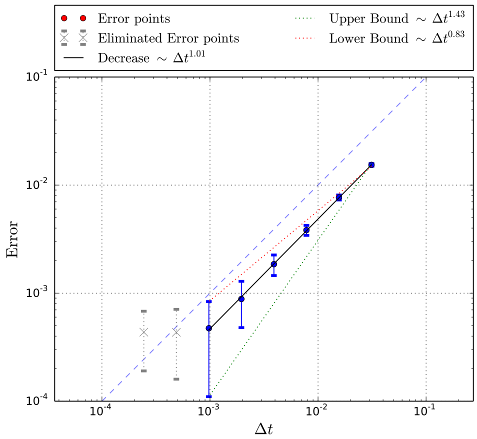
The red dotted line connects the low bound of the confidence interval connected to the first data point to the upper bound of the confidence interval of the last data point. This would represent crudely a lower bound of the slope. Reciprocally, the green dotted line connects the upper bound of the confidence interval of the first data point to the lower bound of the confidence interval of the last point. This would represent an upper bound of the slope.

Having two estimates of the slope can help in different cases. If the size of confidence interval is large compared to the error, then the position of the data point inside the confidence interval can have a significant change on the slope. It is in such cases that a bound based on the confidence interval is useful. A weighted linear regression might be useful, but it is more complicated when we have few data points, as it is the case in our situation.

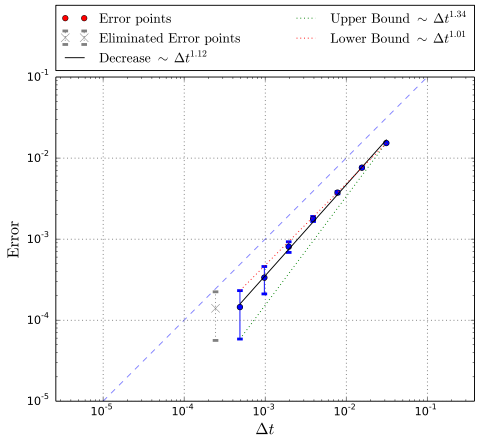
If the size of the confidence interval is small compared to the values of the data, then the linear regression is sufficiently stable as the data points do not have too much uncertainty in their values.

It can be seen that for very small values of Δt the statistical error dominates so these points will be eliminated in the estimation of the convergence rate (these are shown as the red cells in table 14 in the Appendix). Another useful result is the variance and confidence interval for only one of the N_{Err} estimators of $\mathbb{E}f(\bar{X}_T, \bar{U}_T)$ in (2.4). These results are presented in table 12 in the Appendix. These confidence intervals present an approximate cut-off range of values useful when eliminating simulation points from $\overline{\overline{\text{Error}}}[\cdot]$. It can be noticed that the variances for the case $x_0 = 0.1, u_0 = -1.1$ are larger than those of the other test cases, thus the statistical error is also larger and more points need to be eliminated in the first case compared to the other two.

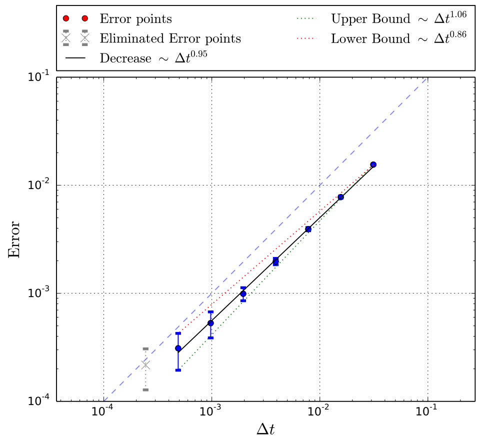
It can be seen that in all cases, this rate of convergence is close to the theoretical linear value presented in Theorem 1.6.



(a) $x_0 = 0.1,\ u_0 = -1.1,$ $\overline{\overline{\text{Error}}}[f] \sim \Delta t^{1.01}$



(b) $x_0 = 0.01,\ u_0 = -0.11,$ $\overline{\overline{\text{Error}}}[f] \sim \Delta t^{1.12}$



(c) $x_0 = 0.001,\ u_0 = -0.011,$ $\overline{\overline{\text{Error}}}[f] \sim \Delta t^{0.95}$

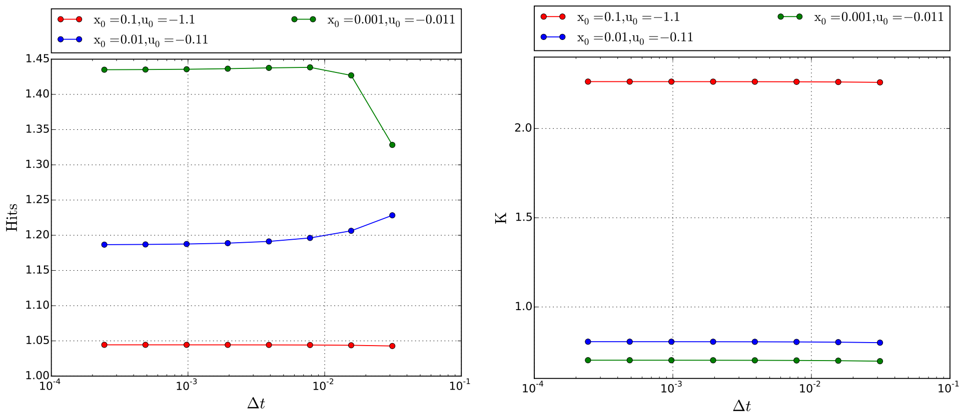
Figure 2.1: Error convergence estimates in the case of $b \equiv 0$

In table 2.1, we give the upper and lower bounds obtained through our method. We also present the result of the simple linear regression: the obtained slope and the $p-$value by testing the estimated slope against the theoretical value of 1. This $p-$value is underestimated because it only considers the errors between the linear estimation and the data points without considering that the data points are in a confidence interval.

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	1.43	1.34	1.06
Lower Bound	0.83	1.01	0.86
OLS slope Estimation	1.01	1.12	0.95
$p-$value	0.1104	0.0013	0.0165

Table 2.1: Result of convergence rate estimation $b = 0$

Finally, we estimate the random variable $\sum_{t \le T} \mathbb{1}_{\bar{X}^c_t = 0}$ which gives the number of times the process $\bar{X}$ hit the reflective boundary and $\bar{K}_T$, the discretised version of K_T defined in (1.1) which gives the value of the process $\bar{U}^c$ when the reflective boundary is hit. These values are presented as functions of the discretisation time-step Δt and are useful as measures of the supposition of only one collision per time step. The results are presented in Figure 2.2 and the confidence intervals in table 13.



(a) Number of reflections

(b) Value of $\bar{K}_T$

Figure 2.2: Statistics in the case of $b \equiv 0$ (Δt decreases right to left)

We can see that as the point (x_0, u_0) approaches the origin, the number of times a trajectory reflects increases, while the velocity at which the reflected boundary is hit decreases. This is fairly straightforward, since if a particle hits the boundary with high velocity, then the reflected velocity will be reversed but still maintain a high magnitude, meaning that the reflected particle goes quickly in the opposite direction of the wall and the probability of it returning is decreased. Also, fairly quickly, the number of reflections and $\bar{K}_T$ become constant, meaning that the supposition of one collision per time step is verified in this case because a decrease in time step does not modify the number of collisions. Another result that stands out is the fact that $\bar{K}_T$ seems much less sensitive to the variation of Δt than the number of hits, meaning that even if errors are committed while estimating the factor $\sum_{t \leq T} \mathbb{1}_{\bar{X}^c_t = 0}$, the weighted sum $\sum_{t \leq T} \bar{U}^c_t \mathbb{1}_{\bar{X}^c_t = 0}$ is still valid.

Richardson-Romberg output

We denote by $\hat{f}(\bar{X}_T^{(\Delta t)}, \bar{U}_T^{(\Delta t)})$ the estimator of $\mathbb{E}f(\bar{X}^c_T, \bar{U}^c_T)$ so that (2.4) can be rewritten as:

$$\overline{\overline{\text{Error}}}[f](\Delta t)^2 := \frac{1}{N_{\overline{\overline{\text{Err}}}}} \sum_{i=0}^{N_{\overline{\overline{\text{Err}}}}} \left(\mathbb{E}f(X^c_T, U^c_T) - \hat{f}(\bar{X}^{(\Delta t)}, \bar{U}^{(\Delta t)}) \right)^2 .$$

Then we can introduce the Richardson-Romberg estimator and the equivalent error:

$$\overline{\overline{\text{ErrorRR}}}[f](\Delta t)^2 := \frac{1}{N_{\overline{\overline{\text{Err}}}}} \sum_{i=0}^{N_{\overline{\overline{\text{Err}}}}} \left(\mathbb{E}f(X^c_T, U^c_T) - \left(2\hat{f}(\bar{X}_T^{(\Delta t/2)}, \bar{U}_T^{(\Delta t/2)}) - \hat{f}(\bar{X}_T^{(\Delta t)}, \bar{U}_T^{(\Delta t)}) \right) \right)^2 . \quad (2.5)$$

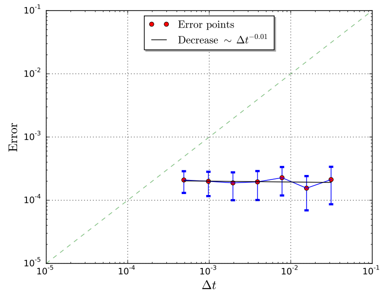
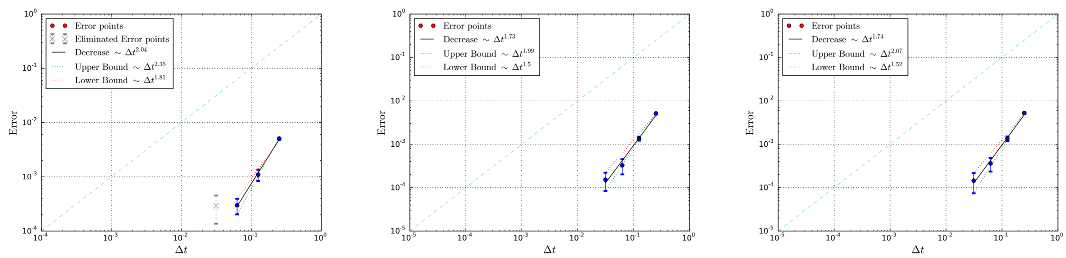


Figure 2.3: $\overline{\overline{\text{ErrorRR}}}[f]$ in the case of $b \equiv 0$ and $x_0 \;=\; 0.01,\; u_0 = -0.01$

We noticed that for the Brownian and $\sin$ cases, the Richardson-Romberg estimator converges very quickly for the set of time steps considered up to the present moment. For example, only the statistical error can be seen in figure (2.3). Therefore, coarser time-steps are needed, so the time steps $\Delta t = \{2^{-5}, \ldots, 2^{-2}\}$ were considered. The results are presented in the log-log plots 2.4 and, more precisely, in the table 2.2. For the estimation, we removed the parts of the curve that are flat (marked in red in table 2.2), which represent the statistical error and not the bias reduction.



(a) $x_0 = 0.1,\; u_0 = -1.1$, $\overline{\text{ErrorRR}} \sim \Delta t^{2.06}$

(b) $x_0 = 0.01,\; u_0 = -0.01$, $\overline{\text{ErrorRR}} \sim \Delta t^{1.73}$

(c) $x_0 = 0.001,\; u_0 = -0.011$, $\overline{\text{ErrorRR}} \sim \Delta t^{1.74}$

Figure 2.4: Error convergence estimates in the case of $b \equiv 0$

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	2.35	1.99	2.07
Lower Bound	1.81	1.5	1.52
OLS slope Estimation	2.04	1.73	1.74
$p-$value	0.37	0.11	0.07

Table 2.2: Result of convergence rate $\overline{\overline{\text{ErrorRR}}}[f]$ for $b \equiv 0$

It can be seen that it is difficult in the case of the Richardson Romberg error since there are quite few data points but the numerical estimate seems to be in the interval $[1.5, 2]$.

$\sin$ function case

In this test case and in the following in subsubsection 2.3 it is not possible to calculate an analytic value for the reference result $\mathbb{E}h(X_T^c, U_T^c)$ with h any test function. Two options are considered in this case. The first one is using a PDE approach.

PDE approach

Assume ψ is a smooth enough test function and consider the function defined for any $(t,x,u) \in Q_T$ as $F(t,x,u) = \mathbb{E}\psi(X_T^{t,x,u}, U_T^{t,x,u})$ where $(X_t^{0,x,u}, U_t^{0,x,u})_{0\le t\le T}$ is a solution to the equation (1.1). Then F is a weak solution to

$$\begin{cases} \partial_t F + (u \cdot \nabla_x F) + (b(x,u) \cdot \nabla_u F) + \dfrac{\sigma^2}{2} \triangle_u F = 0, \text{ on } Q_T, \\ F(T,x,u) = \psi(x,u), \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ F(t,x,u) = F(t,x,u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+ . \end{cases} \tag{2.6}$$

Since the techniques that will be used to numerically solve the previous PDE involve spatial discretisation and the domain $\mathcal{D} \times \mathbb{R}^d$ is infinite, we have imposed different boundary conditions in order to obtain a bounded domain. Coming back to our case, we have that $d \equiv 1$, so the domain of the PDE is $(0,+\infty) \times \mathbb{R}$. We impose:

- periodic boundary conditions on the x coordinate at x_p.
- Dirichlet boundary conditions on the u coordinate at $\pm u_D$.

and we denote as Ω the domain that is obtained.

So, the PDE (2.6) is transformed into

$$\begin{cases} \partial_t F + (u \cdot \partial_x F) + b(x,u)\partial_u F + \dfrac{\sigma^2}{2} \partial_{uu} F = 0, \text{ on } (0,T) \times (0,x_p) \times (-u_D, u_D), \\ F(T,x,u) = \psi(x,u), \text{ on } (0,x_p) \times (-u_D,u_D), \\ F(t,0,u) = F(t,0,-u), \text{ on } (0,T) \times (-u_D,u_D), \\ F(t,x,u_D) = F(t,x,-u_D) = 0, \text{ on } (0,T) \times (0,x_p), \\ F(t,0,u) = F(t,x_p,u), \text{ on } (0,T) \times (-u_D,u_D)\,, \end{cases} \tag{2.7}$$

where $F(t,x,u) = \psi(\widetilde{X}_T^{t,x,u}, \widetilde{U}_T^{t,x,u}) \mathbb{1}_{T \le \tau_{\pm u_D}}$ where $\tau_{\pm u_D} = \inf\left\{ t \ge 0 \,\middle|\, |\widetilde{U}_t^{0,x,u}| = u_D \right\}$ and $(\widetilde{X}_t^{0,x,u}, \widetilde{U}_t^{0,x,u})_{t\ge 0}$ solves the SDE

$$\begin{cases} \widetilde{X}_t = \left[x + \displaystyle\int_0^t \widetilde{U}_s \, ds \right] \mod x_p \\ \widetilde{U}_t = u + \displaystyle\int_0^t b(\widetilde{X}_s, \widetilde{U}_s)\, ds + \sigma W_t - \sum_{0<s\le t} 2\widetilde{U}_{s^-} \mathbb{1}_{\widetilde{X}_s = 0} \,. \end{cases} \tag{2.8}$$

We consider that $x_p = 1$. Also, by [Gobet, 2000], it is known that in the case of killed diffusions, for discretised schemes, the error converges as the square root of the time-step. Since this is below the predicted convergence rate for our scheme that is linear, in order not to bias the results, u_D was chosen sufficiently large such that very few simulated trajectories actually hit the absorption border. So $u_D = 10$.

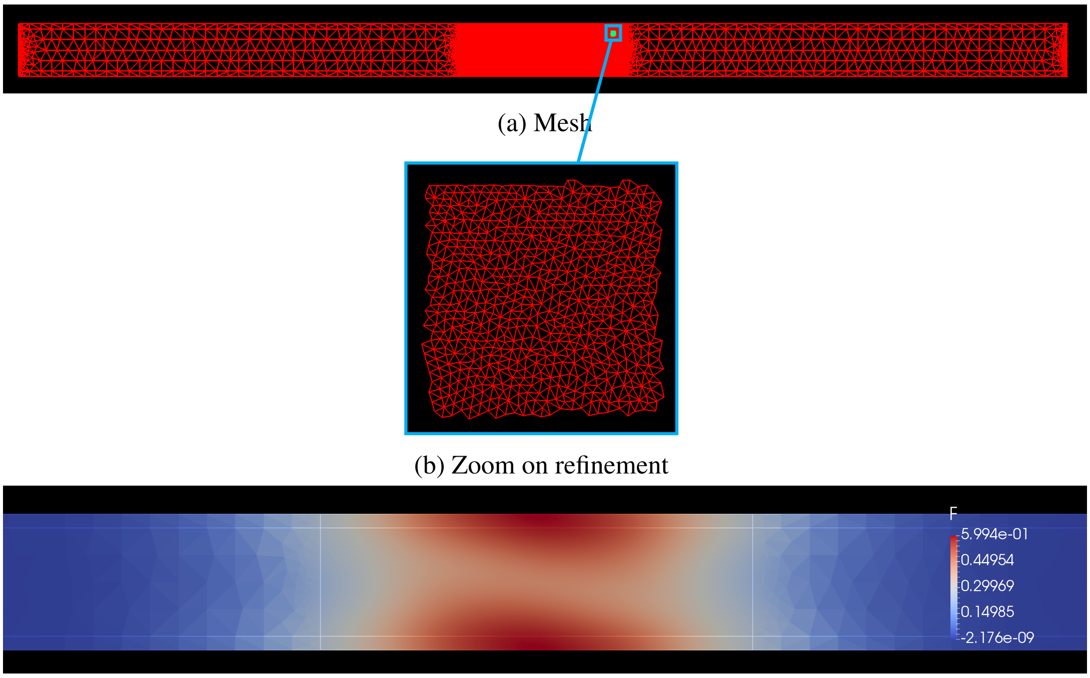
We consider the test function to be $g\colon (x,u) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \exp\left(-(x-0.5)^2 - u^2\right)$ since numerically it is basically on a compact support in the u axis, while respecting needed periodic conditions:

- specular reflection at $x = 0$: $g(0,-u) = g(0,u)$
- particle moving on a torus so $g(1,u) = g(0,u)$, for $u \ge 0$.

Since our reference result is obtained through a numerical it contains discretisation errors that we will try to crudely estimate.

Output with PDE reference result

The numerical reference result was obtained by using the software FreeFem++ [Hecht, 2012] involving a finite element method. Due to the shape of the domain (very elongated in the u coordinate) and to the fact that the terminal function g is very flat outside the central lobe, we have refined the area close to the centre, the region $[-1, 1]$. We show in figure 2.5 the mesh and the numerical result obtained.



(a) Mesh

(b) Zoom on refinement

(c) Zoom on interest region of the solution

Figure 2.5: On the numerical solution of the PDE (2.8)

The parameters of the discretisation are:

- refined centre rectangle size of 2×1 in the (u, x) coordinate: 1000×1000 discretisation points
- outer rectangles each of size 8×1 in the (u, x) coordinate: 40×80 discretisation points
- time discretisation : $\Delta t = 10^{-4}$.

Since in $b \equiv \sin$ and $b\colon (u) \mapsto -5(u + 5)$ cases, we replace the analytic reference result in $\overline{\overline{\text{Error}}}$ by a PDE result, any error on this PDE result creates a bias in the estimation of the convergence rate. There are very few results that involve a priori estimations concerning parabolic PDEs with specular boundary conditions. One is in the case of Dirichlet equation in dimension $d = 2$ with solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and u_h its approximation on a regular mesh with P^1 finite elements. Then there exists a constant C such that we have the bounds

$$\|u - u_h\|_{H^m(\Omega)} \leq Ch^{2-m} \|u\|_{H^2} .$$

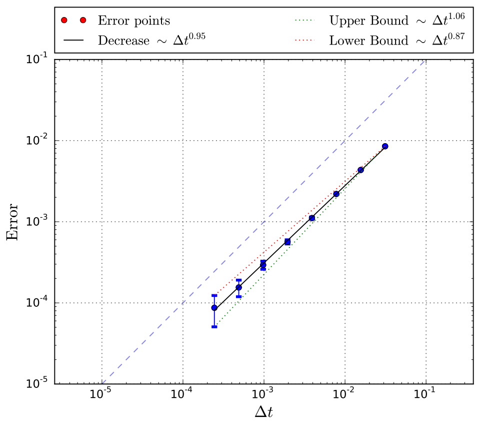
It has been shown in Theorem 1.6 that the solution of our PDE is indeed in H^2. If we consider $m = 1 + \varepsilon$ for any $\varepsilon > 0$ then by Morrey's inequality, $H^{1+\varepsilon}(\mathbb{R}^2) \subset C^{0,\varepsilon}(\mathbb{R}^2)$, where $C^{0,\varepsilon}$ is the $\varepsilon-$Hölder space. And since our domain Ω is bounded, the $\varepsilon-$Hölder space is embedded in $L^\infty(\Omega)$. Therefore, the L^∞ error is bounded by $Ch^{1-\varepsilon} \|u\|_{H^2}$, for any $\varepsilon > 0$.

Of course, besides Dirichlet boundary conditions, we also have specular boundary conditions and the problem being considered is parabolic. For the implicit Euler scheme utilised here, the time discretisation error is linear in Δt. We denote $F_h^{\Delta t}$ the numerical solution of the discretised version of the PDE 2.7 and for any $\varepsilon > 0$ there exists two constants C_1 and C_2, such that for and any $(x, u) \in \Omega$:

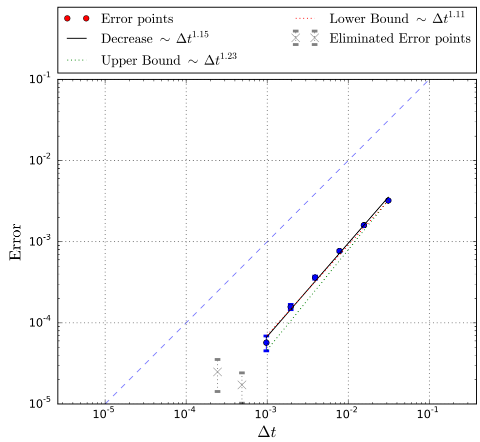
$$|F(0, x, u) - F_h^{\Delta t}(0, x, u)| \leq C_1 h^{1-\varepsilon} + C_2 \Delta t\,.$$

Since $\Delta t = 10^{-4}$ and $h \approx 10^{-3}$, we cannot expect to obtain a better precision than $10^{-4}, 10^{-3}$ depending on the bounding constants.

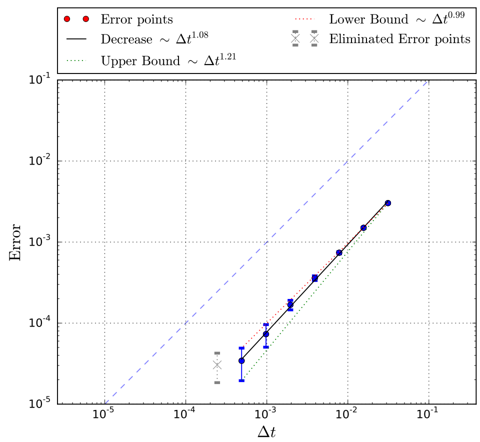
And the results obtained are shown in the log-log plot 2.6 and presented fully in table 17. In table 16, we present the results for one of the $N_{\overline{\overline{\text{Err}}}}$ estimators of $\mathbb{E}g(\bar{X}_T^c, \bar{U}_T^c)$. These values can be useful when determining a cut-off value in order to eliminate data points where the statistical error dominates.



(a) $x_0 = 0.1,\ u_0 = -1.1,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{0.95}$



(b) $x_0 = 0.01,\ u_0 = -0.01,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{1.23}$



(c) $x_0 = 0.001,\ u_0 = -0.011,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{1.08}$

Figure 2.6: Error convergence rates in the case of $b \equiv \sin$

Once again we can notice that the estimated slope, which estimates the decrease rate, is close to the theoretical value of 1.

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	1.06	1.23	1.21
Lower Bound	0.87	1.11	0.99
OLS slope Estimation	0.95	1.15	1.08
$p-$value	10^{-3}	0.0045	$6 \cdot 10^{-4}$

Table 2.3: Result of convergence rate $\overline{\overline{\text{Error}}}[g]$ for $b \equiv \sin$

Output with Monte-Carlo reference result

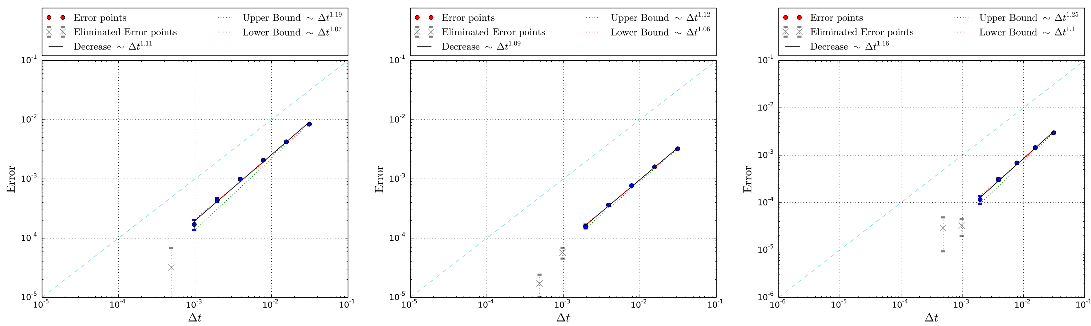
In the previous paragraph, we saw the results when the reference result is obtained from a PDE. In this section we shall consider that the reference result is obtained from an independent Monte Carlo simulation. This Monte Carlo result is calculated on 10^8 trajectories and the time step is the same as the finest trajectory that has been calculated. We remove the results of the finest trajectory from our output.

We present the results obtained for these reference values in table 2.4.

Initial Condition	Result	Variance	$1/2-$Conf Int
$x_0 = 0.1, u_0 = -1.1$	0.385	0.112	$6.71{\cdot}10^{-5}$
$x_0 = 0.01, u_0 = -0.11$	0.597	0.083	$5.76{\cdot}10^{-5}$
$x_0 = 0.001, u_0 = -0.011$	0.599	0.082	$5.73{\cdot}10^{-5}$

Table 2.4: Reference values obtained through MC

Because of the size of the confidence interval of the reference results (of order 10^{-4}), we eliminate all data points smaller than 10^{-4}. In the plots 2.7, we present the results.



(a) $x_0 = 0.1,\ u_0 = -1.1.$ (b) $x_0 = 0.01,\ u_0 = -0.01.$ (c) $x_0 = 0.001,\ u_0 = -0.011.$

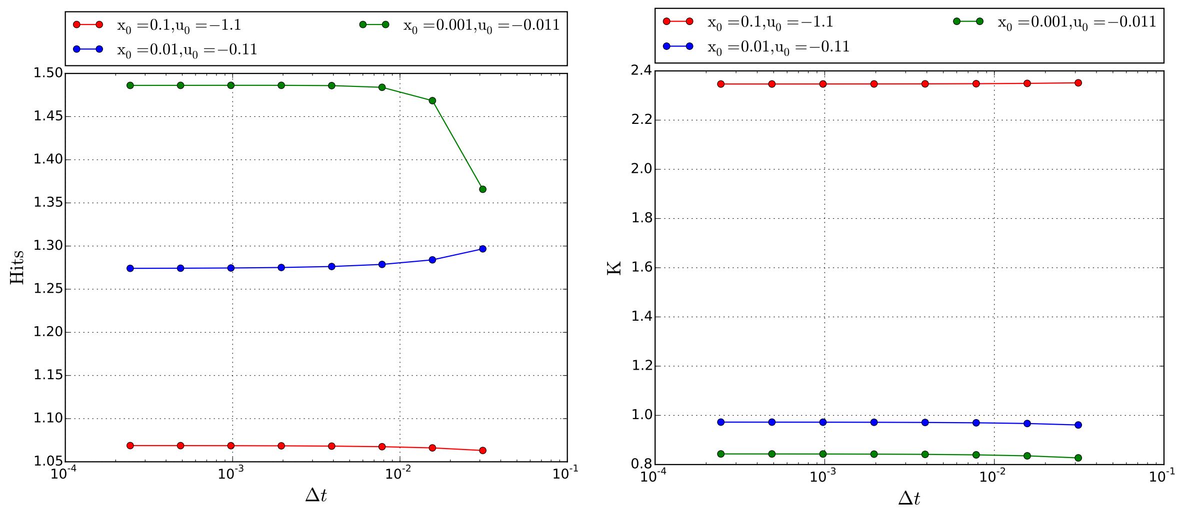
Figure 2.7: $\overline{\overline{\text{Error}}}[g]$ in the case of $b \equiv \sin$ MC reference result

And in table 2.5, we present the estimated values and bounds that were calculated. We can notice that when utilising a finer Monte Carlo reference result, as opposed to a PDE reference result, we have a certain bias to obtain a higher convergence order. But since we have a more precise error for the reference, it is easier to determine a cut-off value.

Finally, we estimate the number of times the process hits the reflective border and the value of $\bar{K}_T$. The same conclusions as the Brownian case apply and it is noticeable that there are few differences between the two test cases. The hypothesis of one collision per time step seems to be verified once more.

Table 2.5: Result of convergence rate $\overline{\overline{\text{Error}}}[g]$ for $b \equiv \sin$ - MC Reference

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	1.19	1.12	1.25
Lower Bound	1.07	1.06	1.11
OLS slope Estimation	1.11	1.09	1.16
$p-$value	0.014	0.12	0.016



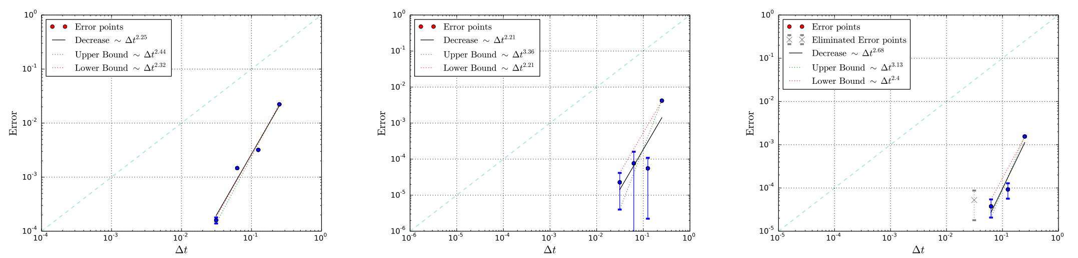
(a) Number of reflections

(b) Value of K_T

Figure 2.8: Statistics in the case of $b \equiv \sin$

Richardson-Romberg extrapolation

In figure 2.9, we plot the results of the Richardon-Romberg convergence rate estimation. In the case of $x_0 = 0.1, u_0 = -1.1$, the decrease is of order $\Delta t^{2.25}$. In the other test cases, the majority of the points lie in the interval $[10^{-5}, 10^{-5}]$. These values are of the order of the confidence intervals in table 18. The reduction of the bias happens very quickly and the statistical error induces too much noise, thus making it difficult to extract any information on the rate of convergence of the error.



(a) $x_0 = 0.1$, $u_0 = -1.1$, $\overline{\text{ErrorRR}}[g] \sim \Delta t^{2.25}$

(b) $x_0 = 0.01$, $u_0 = -0.01$, $\overline{\text{ErrorRR}}[g] \sim \Delta t^{2.11}$

(c) $x_0 = 0.001$, $u_0 = -0.011$, $\overline{\text{ErrorRR}}[g] \sim \Delta t^{2.68}$

Figure 2.9: Richardson-Romberg error convergence estimation in the case of $b \equiv \sin$

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	2.44	3.36	3.13
Lower Bound	2.32	2.21	2.4
OLS slope Estimation	2.25	2.21	2.68
$p-$value	0.022	0.21	0.14

Table 2.6: Result of convergence rate $\overline{\overline{\text{ErrorRR}}}[g]$ for $b \equiv \sin$ (values in red as they are not deemed statistically significant)

Ornstein-Uhlenbeck case

We now consider the drift $b\colon (x, u) \mapsto -5(u + 5)$. This does not respect the condition on the drift in order to prove our theorem, but we can show some numerical results. As in the previous case, with a sinusoidal drift term, we do not have an explicit analytic solution, so one must be calculated. We consider a reference result obtained by a PDE approach and, afterwards, a Monte-Carlo reference result will be analysed.

Output with PDE reference result

We shall consider the same SDE (2.8) and corresponding PDE at (2.7) with the appropriate change in the drift b.

Due to the nature of the chosen Ornstein-Uhlenbeck drift component that presents a strong mean reversion term, there is a strong transport term that makes the analysis more complicated.

For this PDE solver, we considered an implicit finite difference scheme with up-wind discretisation on a uniform grid. Let $F_{\Delta t, h_x, h_u}$ be the numerical solution obtained. Then, we have that there exists three values C_1, C_2 and C_3 that depend on the regularity the exact solution of (2.7) such that for any $(x, u) \in \Omega$:

$$|F(0, x, u) - F_{\Delta t, h_x, h_u}(0, x, u)| \leq C_1 \Delta t + C_2 h_x + C_3 h_u^2 \,,$$

where Δt is the time discretisation, h_x the $x-$axis and h_u the $u-$axis discretisation parameters.

The parameters of discretisation are:

- $\Delta t = 10^{-4}$
- $h_x = 10^{-4}$
- $h_u = 10^{-2}$

meaning that the discretisation error for an usual parabolic equation would be of order 10^{-4}.

We plot the solution obtained by the PDE simulation:

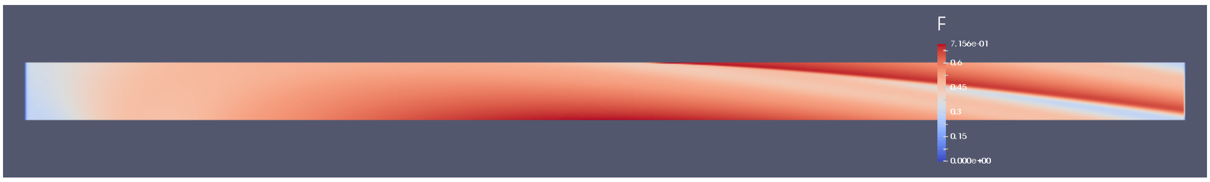
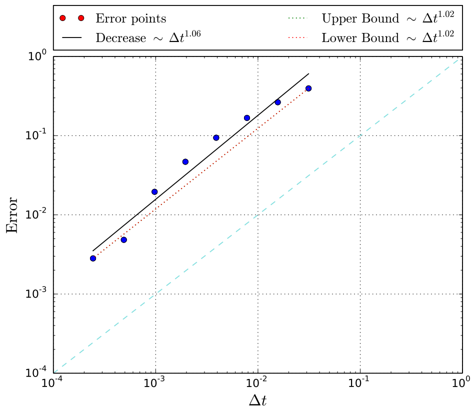


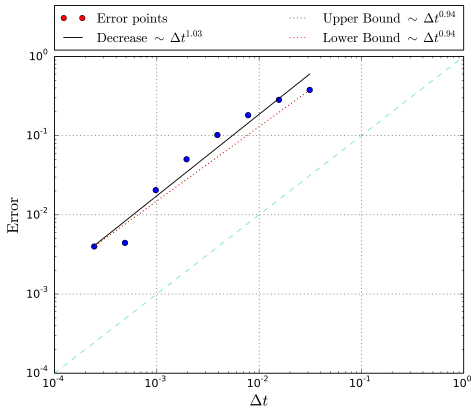
Figure 2.10: PDE solution for the Ornstein-Uhlenbeck case

We can notice that there is an impact of the stiffness of the drift compared to the $\sin$ case. A transport, wave front pattern appears in the solution.

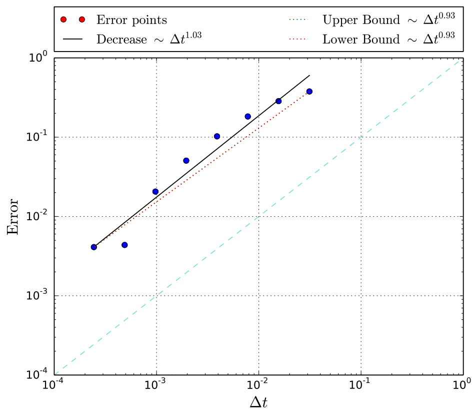
In the plots 2.11, we show the obtained results and in table 2.7 the estimated rates of convergence. It can be seen that compared to the previous cases, the errors are much larger (at least one order of magnitude). Because of this, the confidence intervals are no longer visible on the graph, even though as seen in table 19, they are of the same order as previous examples. Also compared to the previous solutions, there seems to be a larger dispersion of the points, making fitting to a line more complicated. In the three test cases though presented though, our theorem that gives at least a decrease in Δt is confirmed numerically.



(a) $x_0 = 0.1,\ u_0 = -1.1,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{1.06}$



(b) $x_0 = 0.01,\ u_0 = -0.01,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{1.03}$



(c) $x_0 = 0.001,\ u_0 = -0.011,$ $\overline{\overline{\text{Error}}}[g] \sim \Delta t^{1.03}$

Figure 2.11: Weak error convergence estimations the case of Ornstein Uhlenbeck

In table 2.7, we present the $p-$value under the hypothesis H_0 that the order of convergence is 1, which are sufficiently large to not reject the original hypothesis.

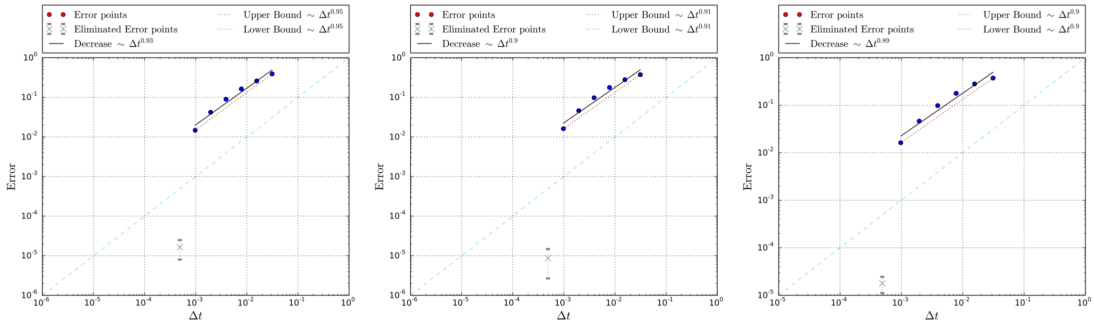
Output with Monte Carlo reference result

In the graph 2.12 we present the results for the weak error calculation when the reference result is taken from a Monte Carlo simulation presented in table 2.8. While the $p-$values from table 2.9 show that we cannot reject the hypothesis H_0 that the convergence rate is linear, we can see that in this

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	1.02	0.94	0.93
Lower Bound	1.02	0.94	0.93
OLS slope Estimation	1.03	1.03	1.03
$p-$value	0.24	0.38	0.39

Table 2.7: Result of estimated convergence rate for Ornstein Uhlenbeck case

case, the Monte Carlo simulation underestimates the value of the convergence rate, which is an opposite behaviour of the $\sin$ case, where the convergence rate was overestimated.



(a) $x_0 = 0.1,\ u_0 = -1.1$. (b) $x_0 = 0.01,\ u_0 = -0.01$. (c) $x_0 = 0.001,\ u_0 = -0.011$.

Figure 2.12: $\overline{\overline{\text{Error}}}[g]$ in the case of Ornstein-Uhlenbeck - MC reference result

Initial Condition	Result	Variance	$1/2-$Conf Int
$x_0 = 0.1, u_0 = -1.1$	0.672	0.0154	$2.48 \cdot 10^{-5}$
$x_0 = 0.01, u_0 = -0.11$	0.704	0.0088	$1.87 \cdot 10^{-5}$
$x_0 = 0.001, u_0 = -0.011$	0.707	0.0081	$1.81 \cdot 10^{-5}$

Table 2.8: Reference values obtained through MC

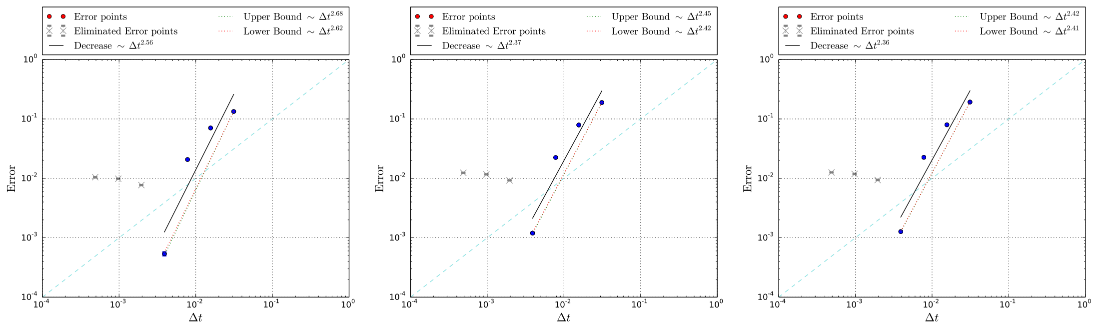
	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	0.95	0.91	0.9
Lower Bound	0.95	0.91	0.9
OLS slope Estimation	0.93	0.9	0.89
$p-$value	0.21	0.16	0.16

Table 2.9: Result of estimated convergence rate for Ornstein Uhlenbeck case - MC reference

Richardson-Romberg extrapolation

We can see that the Richardson Romberg error convergence rates are larger than 2, but the $p-$values in table 2.10 are sufficiently large to not reject the testing hypothesis that the Richardson Romberg convergence rate is statistically different from 2.

It is also possible that the graph points are artificially precise since the error obtained is smaller than the error obtained by the normal weak error estimator.



(a) $x_0 = 0.1,\ u_0 = -1.1$, $\overline{\overline{\text{ErrorRR}}}[g] \sim \Delta t^{2.56}$

(b) $x_0 = 0.01,\ u_0 = -0.01$, $\overline{\overline{\text{ErrorRR}}}[g] \sim \Delta t^{2.37}$

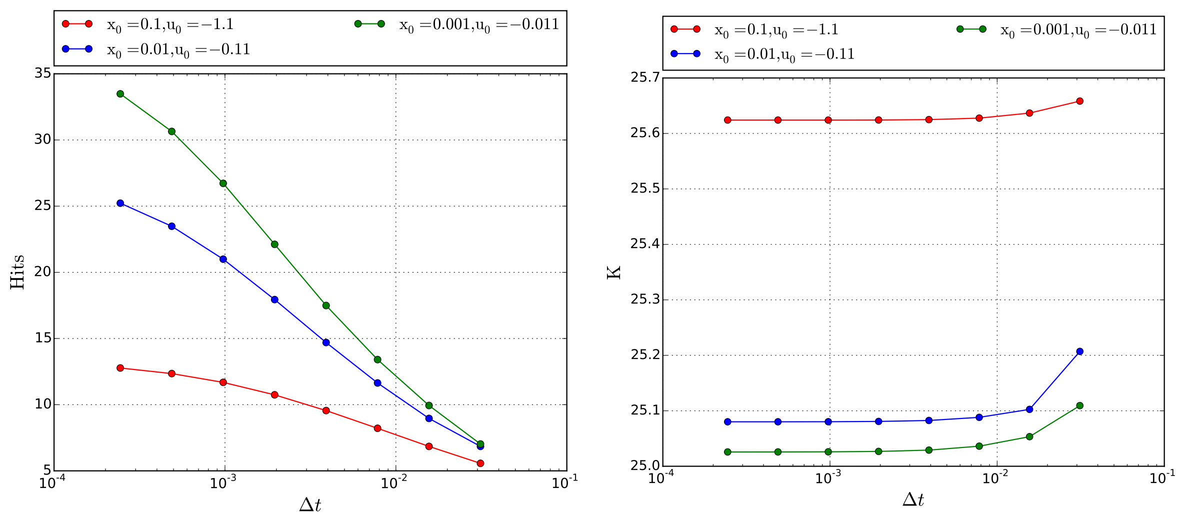
(c) $x_0 = 0.001,\ u_0 = -0.011$, $\overline{\overline{\text{ErrorRR}}}[g] \sim \Delta t^{2.36}$

Figure 2.13: Richardson-Romberg error convergence rates in the case of Ornstein Uhlenbeck

	$x_0 = 0.1, u_0 = -1.1$	$x_0 = 0.01, u_0 = -0.11$	$x_0 = 0.001, u_0 = -0.011$
Upper Bound	2.68	2.45	2.42
Lower Bound	2.62	2.42	2.41
OLS slope Estimation	2.56	2.37	2.36
$p-$value	0.26	0.39	0.4

Table 2.10: Result of Romberg error convergence rate for Ornstein Uhlenbeck case

The plot of the number of hits per time step shows that with the selected range of time steps, we underestimate the number of times the process hits the reflection border. Therefore many of the effects seen in this test case, such as larger dispersion and value of errors, might be due to this underestimation. The value of the velocity process at the collision instants seems to remain fairly flat, implying that while the number of hits is underestimated, the weighted sum K_T over these collision times is estimated fairly well.



(a) Number of reflections

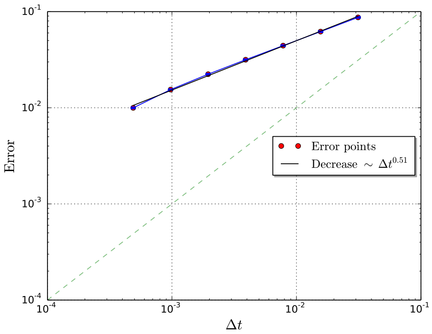
(b) Value of K_T

Figure 2.14: Statistics in the case of Ornstein Uhlenbeck

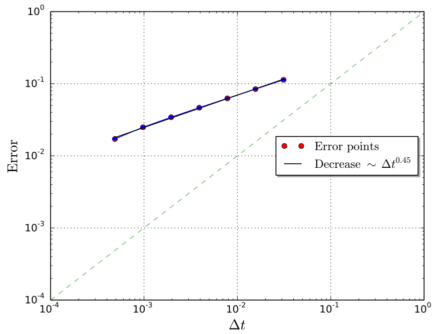
2.4 Strong Error

We also compute the numerical strong error that the symmetrised scheme produces. There is currently no estimate or actual proof that the proposed scheme converges in the strong sense but we consider that it may offer some insights that can complement the numerical study of the weak error.

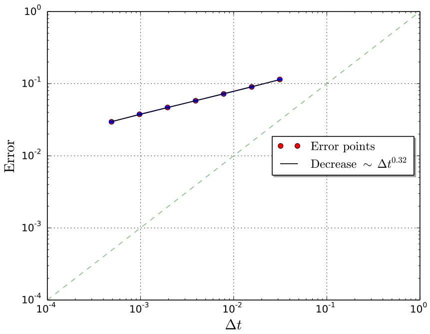
The strong error takes the form $\mathbb{E}\left[\sup_{i=\{1,\ldots,n\}}\left|X^{\text{ref}}_{t_i}-\bar{X}_{t_i}\right|\right]$ on the position and $\mathbb{E}\left[\left|U^{\text{ref}}_T-\bar{U}_T\right|\right]$ for the velocity since it presents discontinuities, where $(X^{\text{ref}},U^{\text{ref}})_{0\le t\le T}$ represent a strong solution of the equation (1.1), which exists by [Bossy and Jabir, 2011] by proving weak existence and pathwise uniqueness. Since it is not possible to exhibit such a function, we replace this process with a version of our scheme calculated with a very fine time-step, denoted as $(\bar{X}^{\text{ref}},\bar{U}^{\text{ref}})_{0\le t\le T}$. The trajectories of the process which we compare to this reference solution, use the Brownian noise as the reference. We shall plot the curves $\Delta t\mapsto\frac{1}{N_{\text{str}}}\sum_{k=1}^{N_{\text{str}}}\left(\sup_{i=\{1,\ldots,n\}}\left|\bar{X}^{\text{ref},k}_{t_i}-\bar{X}^{\Delta t,k}_{t_i}\right|\right)$ and $\Delta t\mapsto\frac{1}{N_{\text{str}}}\sum_{k=1}^{N_{\text{str}}}\left(\left|\bar{U}^{\text{ref},k}_T-\bar{U}^{\Delta t,k}_T\right|\right)$, where $(\bar{X}^{\text{ref},k},\bar{U}^{\text{ref},k})_{0\le t\le T}$ are N_{str} independent realisations of the reference discretised solution and $(\bar{X}^{\Delta t,k},\bar{U}^{\Delta t,k})_{0\le t\le T}$ are solutions obtained with the same Brownian noise and time-step Δt. We present the results obtained for 3 initial values in the log-log plots 2.15 for the Brownian case, 2.16 for the sin case and 2.17 for the Ornstein Uhlenbeck case. In the same plots, the 95% confidence intervals for these Monte Carlo results are plotted, but they are not visible since they are 3 to 5 orders of magnitude smaller than the simulated values, as seen in the tables 19 or 20.



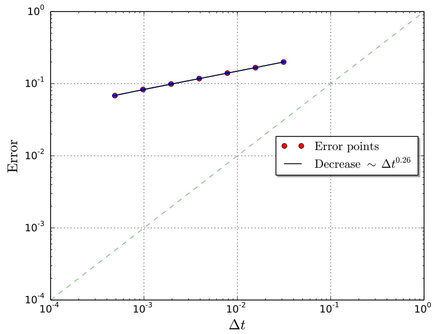
(a) $\bar{X}$ Error $\sim \Delta t^{0.51}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.



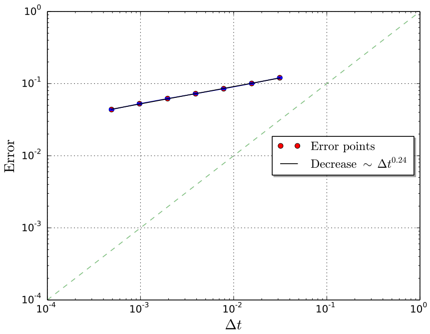
(b) $\bar{U}$ Error $\sim \Delta t^{0.49}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.



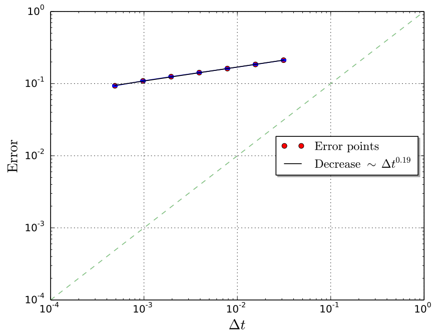
(c) $\bar{X}$ Error $\sim \Delta t^{0.32}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.



(d) $\bar{U}$ Error $\sim \Delta t^{0.26}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.

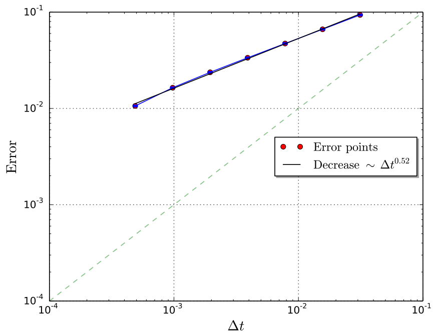


(e) $\bar{X}$ Error $\sim \Delta t^{0.24}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.

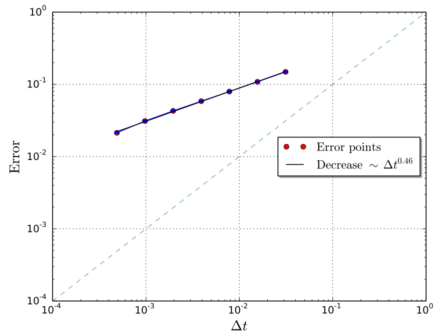


(f) $\bar{U}$ Error $\sim \Delta t^{0.19}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.

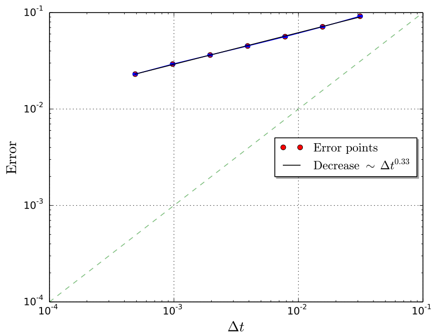
Figure 2.15: Strong Error for $b \equiv 0$



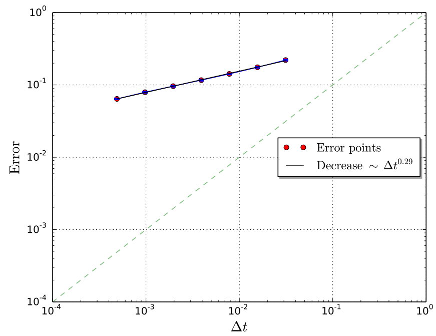
(a) $\bar{X}$ Error $\sim \Delta t^{0.52}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.



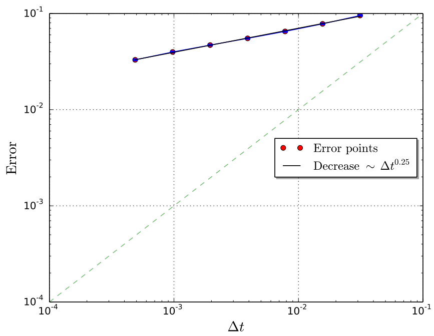
(b) $\bar{U}$ Error $\sim \Delta t^{0.46}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.



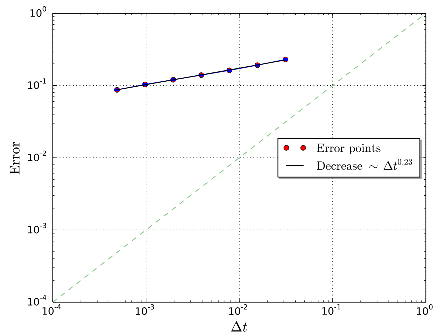
(c) $\bar{X}$ Error $\sim \Delta t^{0.33}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.



(d) $\bar{U}$ Error $\sim \Delta t^{0.29}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.

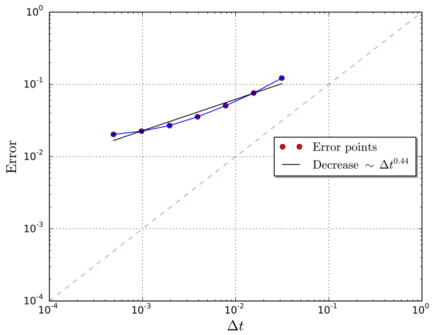


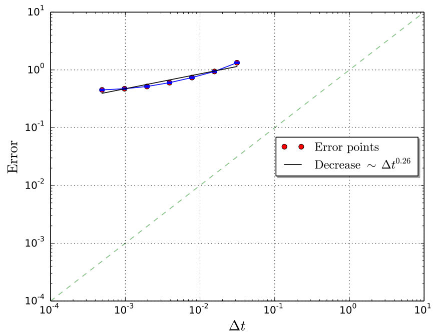
(e) $\bar{X}$ Error $\sim \Delta t^{0.25}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.



(f) $\bar{U}$ Error $\sim \Delta t^{0.23}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.

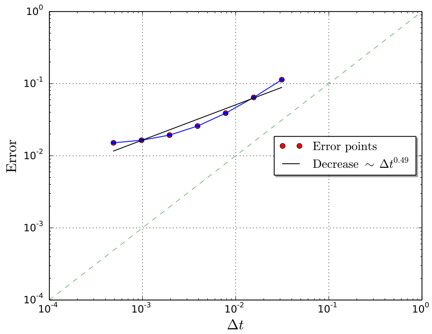
Figure 2.16: Strong Error for $b \equiv \sin$

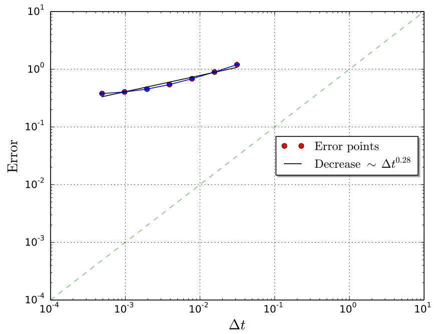




(a) $\bar{X}$ Error $\sim$ $\Delta t^{0.44}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.

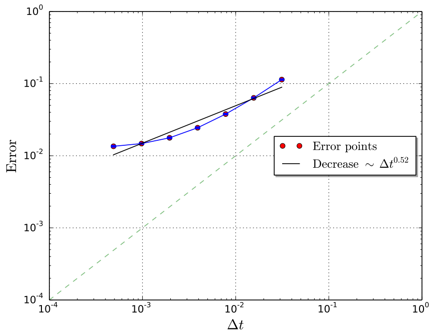
(b) $\bar{U}$ Error $\sim$ $\Delta t^{0.26}$ for initial conditions $x_0 = 0.1,\ u_0 = -1.1$.

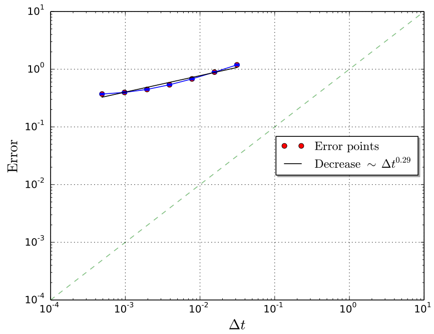




(c) $\bar{X}$ Error $\sim$ $\Delta t^{0.49}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.

(d) $\bar{U}$ Error $\sim$ $\Delta t^{0.29}$ for initial conditions $x_0 = 0.01,\ u_0 = -0.11$.





(e) $\bar{X}$ Error $\sim$ $\Delta t^{0.52}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.

(f) $\bar{U}$ Error $\sim$ $\Delta t^{0.29}$ for initial conditions $x_0 = 0.001,\ u_0 = -0.011$.

Figure 2.17: Strong Error for Ornstein-Uhlenbeck case

Comments

We recall that in [Lépingle, 1995], in the case of a reflected diffusion, the strong error of the scheme is of order $\frac{1}{2}$. Another important fact is that the initial condition is forgotten by the process once the reflective boundary is reached. In the case of specular reflection, at least numerically, the convergence error seems greatly reduced and we can notice also a dependence on the initial values, decreasing as the initial points approach the origin.

Another aspect is that in the Brownian and sinus cases, the curve is quite straight with little deviation between the values as seen in tables 23, 24, 25 and 26. In the case of Ornstein Uhlenbeck drift, we notice larger deviations from the least square line. This can be due to the fact that in the Ornstein Uhlenbeck case, for the selected values of Δt, the hypothesis of one collision per time step is no longer respected. What is less obvious is the convexity of these curves, which imply a overestimation of the strong rate.

And finally, the reduced strong rate convergence will also negatively impact the convergence of a Multi-Level Monte Carlo algorithm.

3 Penalised Schemes

This section contains various tentative numerical experiments regarding penalised schemes that would converge towards a specularly reflected Brownian process.

3.1 Penalisation on the velocity term

In the article [Paoli and Schatzman, 1993], the authors obtain the convergence of a penalised kinetic equation towards a process that admits a specular. Their arguments require Lipschitz conditions on the velocity term, which corresponds to the Brownian term in our case, so a direct extension of their result is not possible. Nevertheless, we consider :

$$\begin{cases} X_t^\lambda = X_0 + \int_0^t U_s^\lambda \, ds \\ U_t^\lambda = U_0 + \int_0^t b(X_s^\lambda, U_s^\lambda) \, ds + \frac{1}{\lambda} \int_0^t (X_s^\lambda)_- \, ds + \sigma W_t \end{cases} \tag{3.1}$$

Due to the fact that the process present a very stiff term, the discretisation of the scheme involved a Runge-Kutta scheme of order m taken from [Abdulle and Li, 2008]. Since we do not know what optimal values of λ to select in order to minimise the error, we assume that we can write

$$\lambda = (\Delta t)^{1+\alpha}$$

and simulate the scheme for α taken in a range of values. For the figures 2.18 and 2.19, we select $\alpha = \{-0.5, -0.4, \cdots, 1.4\}$ while for figure 2.20 we have selected the range $\alpha = \{-0.2, 0, 0.2, \cdots 1.2\}$.

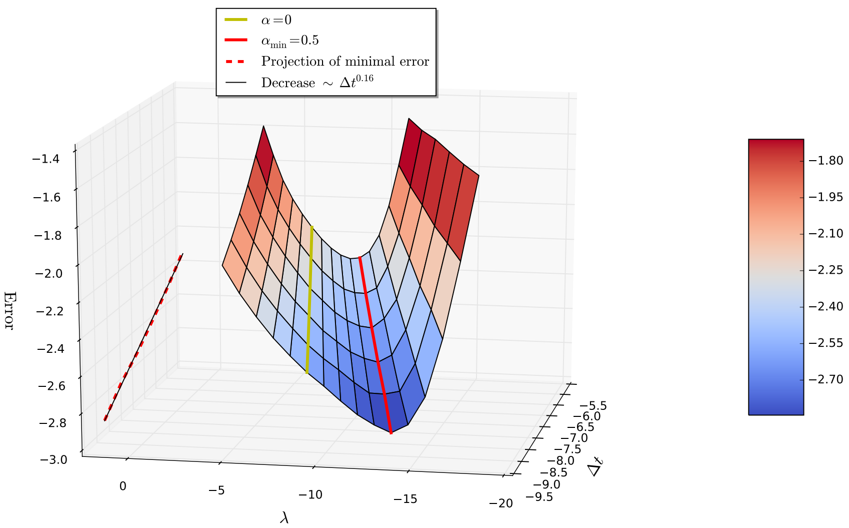


Figure 2.18: Selecting λ for a strong error on $\bar{X}$

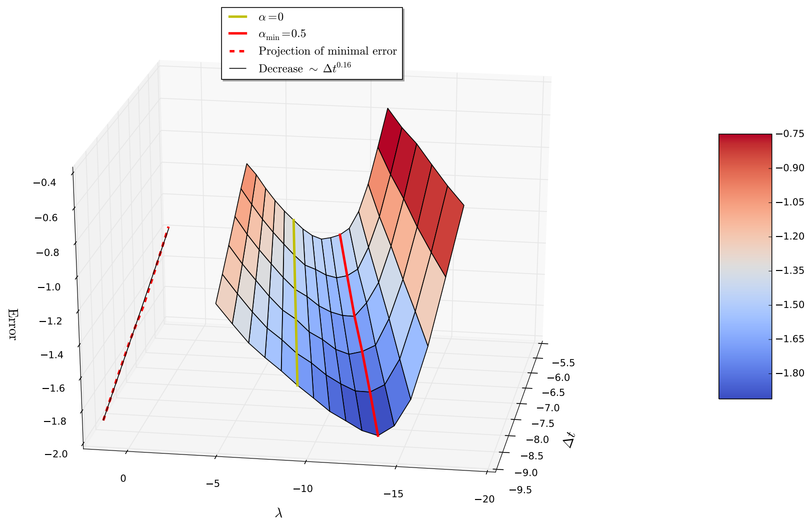


Figure 2.19: Selecting λ for a strong error $\bar{U}$

Concerning the weak error obtained for 1 million trajectories and test function $f\colon (x,u) \mapsto x^2 + u^2$ we obtain:

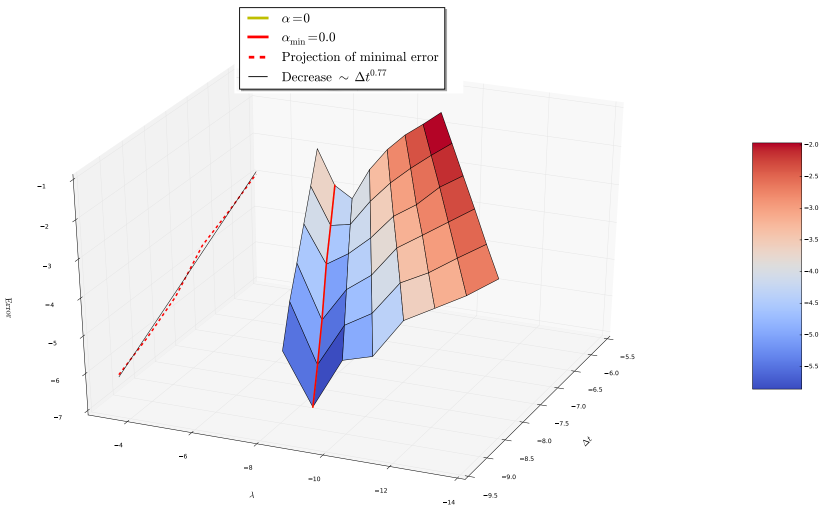


Figure 2.20: Selecting λ for the weak error

The value of α that minimises this error seems to be $\alpha \approx 0$, meaning that $\lambda \approx \Delta t$. The decrease for this selected value is $\Delta t^{0.77}$.

What is noticeable between the weak error and the strong error, is that there are different values of α that minimise the error. For the strong error, it is $\alpha \approx 0.5$, for the weak error $\alpha \approx 0$. If we were to select $\alpha \approx 0.5$ for the weak error, we obtain a decrease in $\Delta t^{0.28}$.

3.2 Reflection: Slominski case

This scheme is based on an article by Slominski [Slominski, 2013] which applies a penalisation on the whole process. In our case, we shall penalise the position to obtain:

$$\begin{cases} X_t^\lambda = X_0 + \displaystyle\int_0^t U_s^\lambda + \frac{1}{\lambda}\int_0^t (X_s^\lambda)_- \, ds \\ U_t^\lambda = U_0 + \displaystyle\int_0^t b(X_s^\lambda, U_s^\lambda)\, ds + \sigma W_t \end{cases} \tag{3.2}$$

In this case, the process $(U_t^\lambda)_{t\in[0,T]}$ no longer represents the velocity of the process $(U_t^\lambda)_{t\in[0,T]}$. In order to simplify calculations, we shall assume that the drift b is $b\colon (x,u) \mapsto b(x)$, thus we can introduce the process $V_t = U_t^\lambda + \dfrac{1}{\lambda}(X_t^\lambda)_-$. Then the process becomes:

$$\begin{cases} X_t^\lambda = X_0 + \displaystyle\int_0^t V_s^\lambda \, ds \\ V_t^\lambda = U_0 + \displaystyle\int_0^t b(X_s^\lambda)\, ds + \sigma W_t + \frac{1}{\lambda}(X_t^\lambda)_- \end{cases} \tag{3.3}$$

A first scheme

$$\begin{cases} \bar{X}_{t_{i+1}}^\lambda = \bar{X}_{t_i}^\lambda + \bar{V}_{t_i}^\lambda \Delta t \\ \bar{V}_{t_{i+1}}^\lambda = b(\bar{X}_{t_i}^\lambda)\Delta t + \sigma(W_{t_{i+1}} - W_{t_i}) + \dfrac{1}{\lambda}\Big((\bar{X}_{t_{i+1}}^\lambda)_- - (\bar{X}_{t_i}^\lambda)_-\Big) \end{cases} \tag{3.4}$$

The results are:

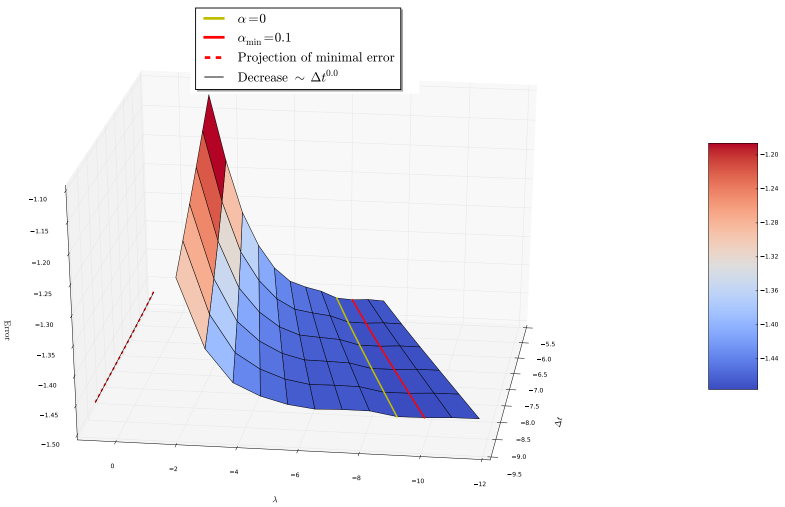


Figure 2.21: Selecting λ for a strong error $\bar{X}$

and

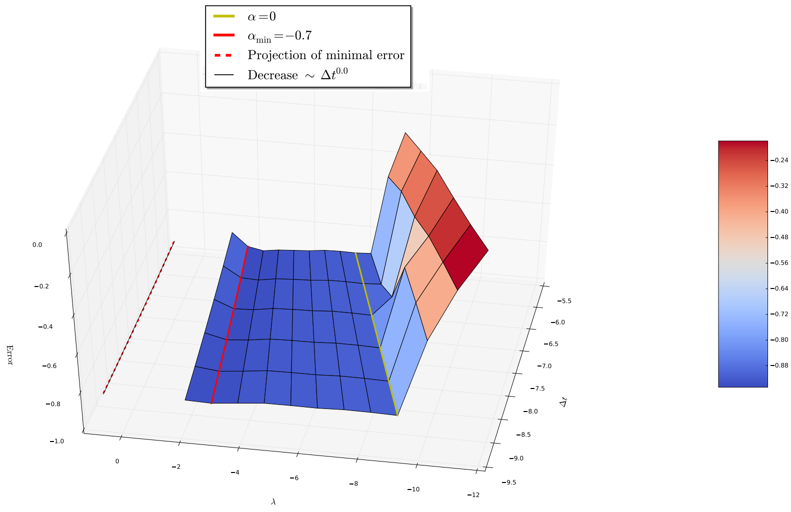


Figure 2.22: Selecting λ for a strong $\bar{U}$

It can be seen that the regions that minimise the error surface are also flat so it is difficult to select a good enough λ.

Rewriting the process

Equation (3.3) can be rewritten as:

$$\begin{cases} X_t^\lambda = X_0 + \int_0^t V_s^\lambda \\ V_t^\lambda = U_0 + \int_0^t b(X_s^\lambda)\, ds + \sigma W_t + \frac{1}{\lambda} \int_0^t \mathbb{1}_{X_s^\lambda \le 0} V_s^\lambda\, ds \end{cases} \tag{3.5}$$

Exponential Scheme

A first scheme that was analysed was a simple discretisation of the position $\bar{X}_{t_{i+1}}^\lambda = \bar{X}_{t_i}^\lambda + \bar{V}_{t_i}^\lambda \Delta t$. Concerning the velocity process, we use an exponential scheme on the velocity during the period the process $(\bar{X}^\lambda)_{0 \le t \le T}$ is in the negative domain.

The results are:

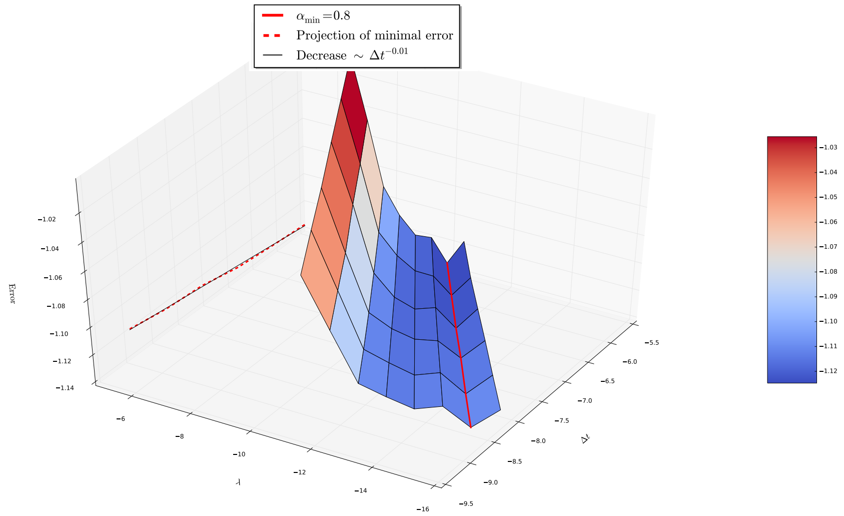


Figure 2.23: Selecting λ for $\bar{X}$

and

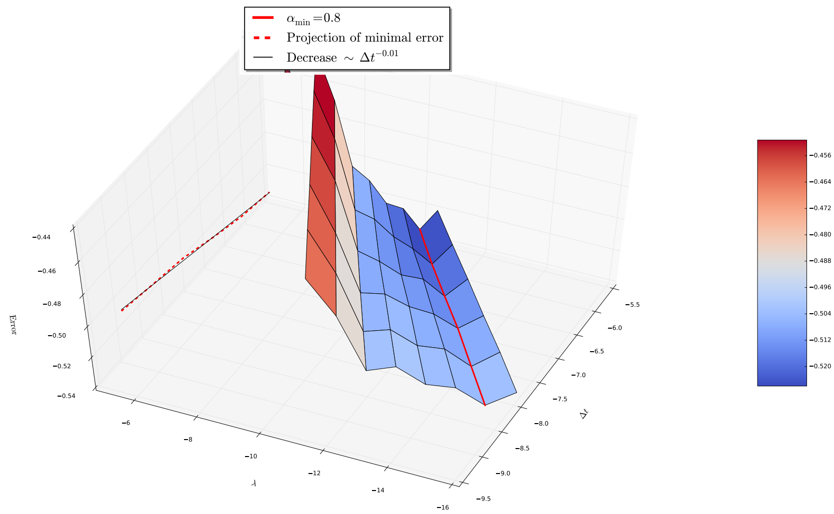


Figure 2.24: Selecting λ for $\bar{U}$

Again the results seem inconclusive. The error does not seem to decrease in Δt, which would not correspond to a normal behaviour.

Retrograde Scheme

A different scheme for (3.5) would be to consider that we penalise the velocity with a value proportional to the velocity needed to enter the negative domain:

$$\begin{cases} \bar{X}^{\lambda}_{t_{i+1}} = \bar{X}^{\lambda}_{t_i} + \bar{V}^{\lambda}_{t_i}\Delta t \\ \bar{V}^{\lambda}_{t_{i+1}} = b(\bar{X}^{\lambda}_{t_i})\Delta t + \sigma(W_{t_{i+1}} - W_{t_i}) + \dfrac{\Delta t}{\lambda}\mathbb{1}_{\bar{X}_{t_i} \leq 0}\bar{V}_{t_{i-1}} \end{cases} \tag{3.6}$$

This gives that:

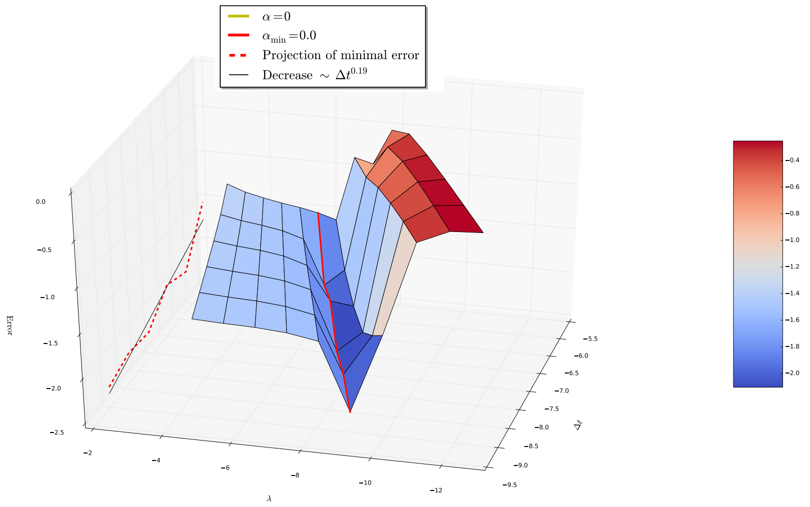


Figure 2.25: Selecting λ for $\bar{X}$

and

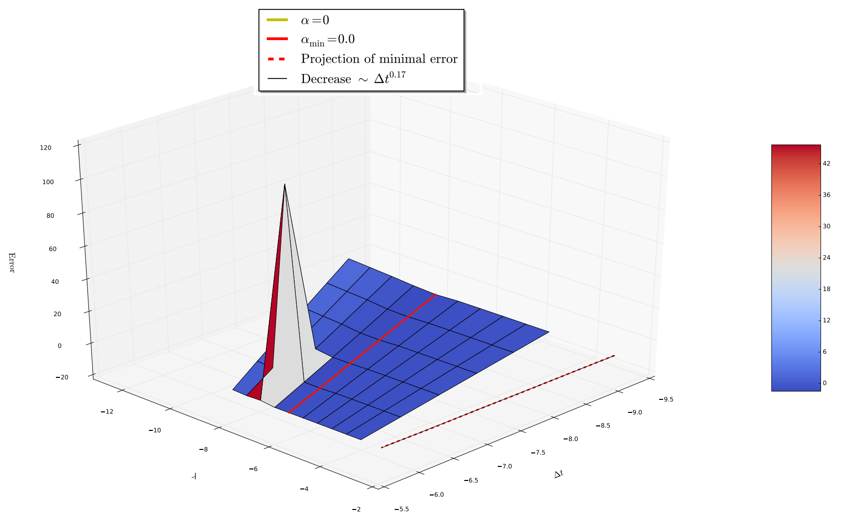


Figure 2.26: Selecting λ for $\bar{U}$

We notice that the scheme has stability issues. Eliminating this instability point:

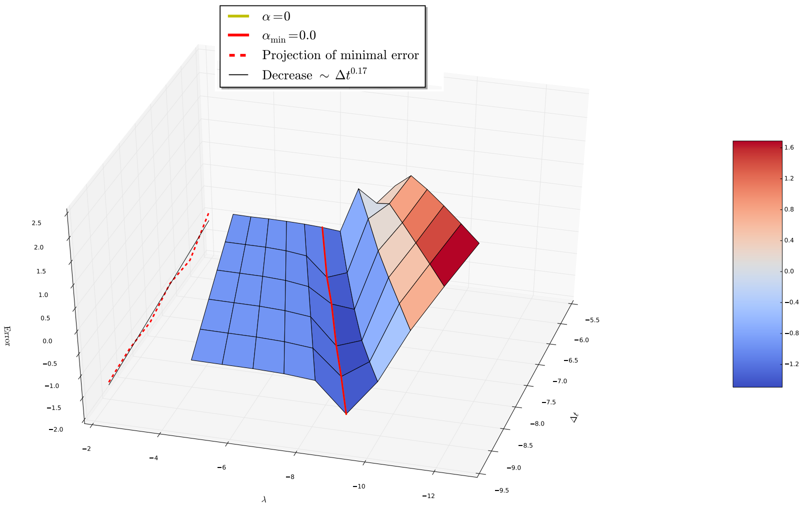


Figure 2.27: Selecting λ for $\bar{U}$

The error is minimised for $\alpha \approx 0$, so $\lambda \approx \Delta t$. The decrease is of order $\Delta t^{0.19}$ for the position and $\Delta t^{0.17}$.

And for the weak error:

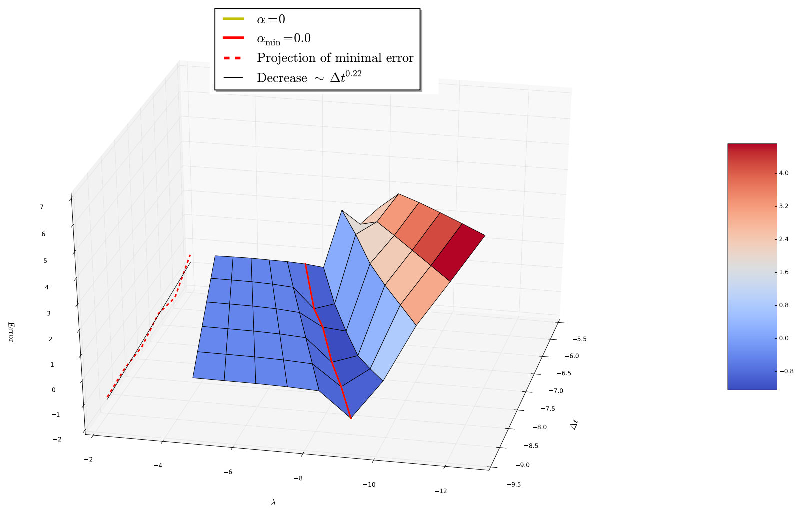


Figure 2.28: Selecting λ for the weak error

The values are $\lambda \approx \Delta t$ and the decrease is of order $\Delta t^{0.22}$.

This scheme seems to behave better compared to the others, but more work is needed to eliminate the instabilities.

4 Multilevel Monte Carlo

In this section we apply the multilevel Monte Carlo method to the symmetrised scheme. The Multilevel Monte Carlo (MLMC) and other similar variance reduction methods appeared independently in different contexts. Heinrich [Heinrich, 1998] developed such a method to estimate integrals that depend on a parameter. Giles [Giles, 2008] extends a previous work by [Kebaier, 2005] to multiple levels. It is his formalism that we present here.

4.1 Formalism

Let:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \tag{4.1}$$

with sufficiently regular coefficients b and σ. Suppose we wish to calculate $\mathbb{E}\bar{X}_T^{(h)}$ where the random variable $\bar{X}_T$ is obtained through a discretisation of an SDE with time step h. The estimator for this expectation is $\widehat{Y} = \frac{1}{N}\sum_{i=1}^{N}\bar{X}_T^{(h,i)}$ where $(\bar{X}_T^{(h,i)})_{i=1,\dots,N}$ are N independent copies of $\bar{X}_T^{(h)}$. We denote by $\bar{P}_l := \bar{X}_T^{(h_l)}$, for time step $h_l = \frac{T}{2^l}$. Then we can write the telescopic sum:

$$\mathbb{E}\bar{P}_L = \mathbb{E}\bar{P}_0 + \sum_{l=1}^{L}\mathbb{E}\left[\bar{P}_l - \bar{P}_{l-1}\right]$$

which we estimate using:

$$\bar{Y}_l = \frac{1}{N_l}\sum_{i=1}^{N_l}\left(\bar{P}_l^{(i)} - \bar{P}_{l-1}^{(i)}\right)$$

The multilevel Monte Carlo method involves selecting an optimal number of levels L and optimal number of simulations N_l for each level l so as to reduce the complexity for a given level of the mean square root error. In order to not increase the variance, for any given level l, $\bar{P}_l^{(i)} - \bar{P}_{l-1}^{(i)}$ is calculated using the same Brownian path, with two different time steps h_l and h_{l-1}. We have then:

Theorem 4.1 (Theorem 3.1 in [Giles, 2008])**.** *Let P denote a functional of the stochastic differential equation (4.1) for a given Brownian path $(W_t)_{t\geq 0}$ an let $\widehat{P}_l$ denote the corresponding time-step $h_l = 2^{-l}T$.*

If there exists independent estimators $\widehat{Y}_l$ based on N_l Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}$, β, c_1, c_2, c_3 such that:

1. $\left|\mathbb{E}\left[\bar{P}_l - P\right]\right| \leq c_1 h_l^{\alpha}$
2. $\mathbb{E}\widehat{Y}_l = \begin{cases} \mathbb{E}\left[\bar{P}_0\right], l = 0 \\ \mathbb{E}\left[\bar{P}_l^{(i)} - \bar{P}_{l-1}^{(i)}\right], l > 0 \end{cases}$
3. $\mathbb{V}ar\left[\widehat{Y}_l\right] \leq c_2 N_l^{-1} h_l^{\beta}$
4. *C_l, the computational complexity of $\widehat{Y}_l$, is bounded by*

$$C_l \leq c_3 N_l h_l^{-1}$$

then there exists a positive constant c_4 such that for any $\varepsilon < e^{-1}$ there are values L and N_l for which the multilevel estimator

$$\widehat{Y} = \sum_{i=0}^{L}\widehat{Y}_l$$

has a mean-square-error with bound

$$MSE := \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}\left[P\right]\right)^2\right] < \varepsilon^2$$

with a computation complexity C with bound

$$C \leq \begin{cases} c_4\varepsilon^{-2}, \beta > 1 \\ c_4\varepsilon^{-2}\left(\log\varepsilon\right)^2, \beta = 1 \\ c_4\varepsilon^{-2-(1-\beta)/\alpha}, 0 < \beta < 1. \end{cases}$$

Cost Analysis

These numerical results are obtained through a standard Multi-level Monte Carlo procedure where the only stopping criterion is when the simulation reaches a maximum number of levels set beforehand. As such, convergence (actual error smaller than target error) is not guaranteed.

Table 2.11: MLMC results for $x_0 = 0.01, u_0 = -0.11$ and $b \equiv 0$

Target error	Actual error	Level	Variance	Trajectories
0.005	0.00017			
		2^{-5}	3.32649	1958471
		2^{-6}	0.190013	349449
		2^{-7}	0.162727	227902
		2^{-8}	0.140153	148778
		2^{-9}	0.12585	99206
		2^{-10}	0.100997	62001
		2^{-11}	0.0925873	41551
		2^{-12}	0.077357	26836
0.001	0.00374			
		2^{-5}	3.31315	48805347
		2^{-6}	0.193624	8811241
		2^{-7}	0.164322	5717492
		2^{-8}	0.141247	3729312
		2^{-9}	0.122799	2445868
		2^{-10}	0.103809	1574088
		2^{-11}	0.0903111	1023944
		2^{-12}	0.0767211	670273

First, it can be seen that the convergence of the algorithm is not always obtained and a better stopping criterion would need to be considered. Also the variance of the different levels does not decrease very fast and as such the number of trajectories on each level also does not decrease very quickly. But since the decrease of the variance is determined by the strong error convergence rate, and this rate is extremely poor, as determined numerically in (2.15c) and (2.15d).

On the other hand, in order to obtain an error of order 10^{-3}, the classical method needed 10^9 trajectories with $\Delta t = 2^{-8}$ as seen in Figure 2.1b while the MLMC method needed at most $5 \cdot 10^7$ trajectories but only for the coarsest level.

Another way to view the gain is to assume that the cost to simulate a trajectory with discretisation step $\dfrac{\Delta t}{2}$ is about twice the cost to simulate a trajectory with discretisation step Δt. We can then consider as a base unit, the cost to simulate one trajectory with discretisation step $\Delta t = 2^{-5}$, which we denote as C_{base}. So the cost to simulate 10^9 trajectories with time step 2^{-8} is

$$C_{\text{classic}} = 10^9 \times \frac{2^{-5}}{2^{-8}} \times C_{\text{base}} = 8 \times 10^9 \times C_{\text{base}}$$

Concerning the Multi-level Monte Carlo, we simulate N_0 base trajectories (i.e. with time step 2^{-5} and for $l > 0$, $N_{l-1} + N_l$ trajectories of time step size $2^{-(l+5)}$, where N_l is the number of trajectories at level l. This gives:

$$C_{\text{MLMC}}^{\text{err}=0.005} \approx 2 \times 10^7 \times C_{\text{base}}$$

and

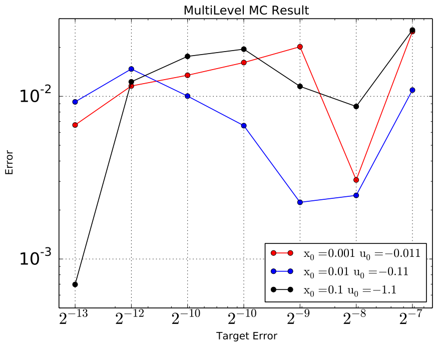
$$C_{\text{MLMC}}^{\text{err}=0.001} \approx 5 \times 10^8 \times C_{\text{base}}$$

so a gain of one to two orders of magnitude in terms of cost for the same target error.

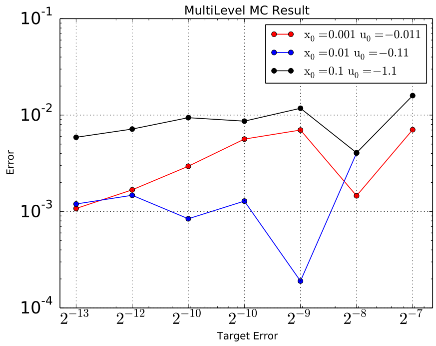
Results

In the log-log plots 2.29 we present the errors between the various multilevel Monte Carlo results and the analytic, in the Brownian case, and PDE reference results. On the ordinate we consider different values for the target error. We do this for different drifts and different initial conditions. It can be seen that these errors do not always decrease as the target error decreases. Also in the Ornstein-Unlenbeck case, the errors seem to increase as the target error decreases, meaning that there is a bias issue.

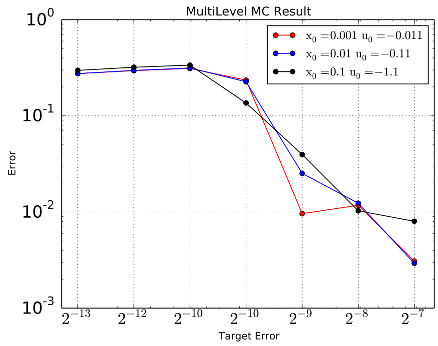
We separate the analysis in two cases: one that contains the drift $b \equiv 0$ and $b \equiv \sin$ and the other being the Ornstein-Uhlenbeck drift. For the first two types of drifts, the results are qualitatively similar, while the Ornstein-Uhlenbeck case seems more problematic.



(a) $b \equiv 0$



(b) $b \equiv \sin$



(c) Ornstein-Uhlenbeck case

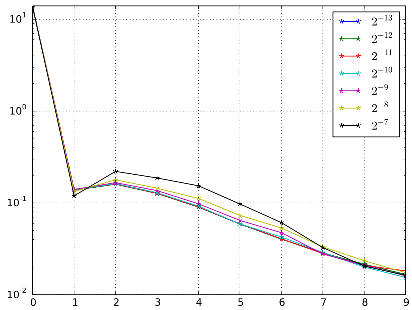
Figure 2.29: Error of the MLMC algorithm

To further analyse, we plot the variance for different levels and different target errors in figures 2.30, 2.31 and 2.32. On the ordinate we have the level (the number of levels are capped at 9), on the abscissae we have the variances while the different curves plotted are the different target errors. The different subfigures are for different initial conditions. In the Appendix, the plots 33, 34 and 35 contain the confidence interval for each level and

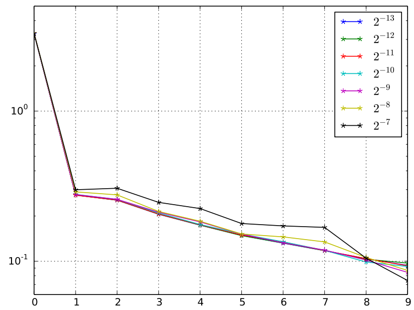
It can be seen that regardless of the chosen target error, the values of the variances remain more or less the same. Also after an initial large decrease, the variances of the different levels seem to converge to 0 at quite a slow pace. After 9 levels, the variance remains of order 10^{-2}, 10^{-3}. The decrease in variance depends on the strong error convergence rate, and we know that for our scheme, the empiric strong error convergence is very slow.

This also implies that the confidence intervals decrease much more slowly and in fact, they are of the same order as the error plotted in 2.29. For example, consider the error for the Brownian in 2.29a with initial conditions $x_0 = 0.01, u_0 = -0.11$. At the beginning for target error 2^{-8}, the error decreases down to order 10^{-3}. Yet in figure 33c of the appendix, we can see that for each level for target error 2^{-8} (in yellow), the confidence interval is also of order 10^{-3}, and if we consider every level independent from the others, we obtain a final confidence interval of size of order 10^{-2}. Thus, the drop seen in the error in figure 2.29 is not very informative since it is of the size of the confidence interval.

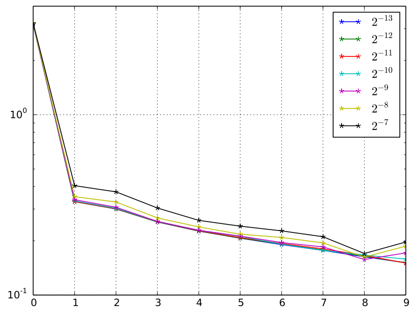
Concerning the Ornstein-Uhlenbeck case, it is difficult to extract any useful information. The results are, qualitatively, very different from the two previous cases. The error seems to increase. The variances plotted in figure 2.32 do not seem to decrease significantly and they also present a concave shape which is in stark contradiction with the expected results. The bias of this scheme seems very significant.



(a) $x_0 = 0.1, \; u_0 = -1.1$



(b) $x_0 = 0.01, \; u_0 = -0.11$



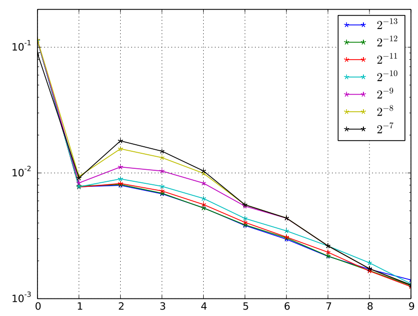
(c) $x_0 = 0.001, \; u_0 = -0.011$

Figure 2.30: Variances $b \equiv 0$

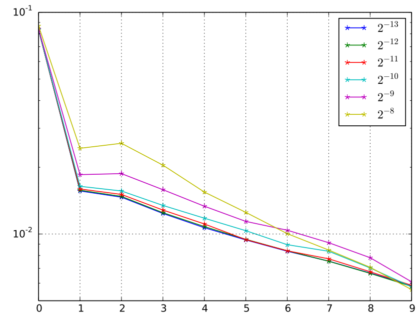
5 Conclusion and perspectives

We have seen in our numerical cases that the weak error does seem to respect our theorem in the series of test cases, more so, we obtain a linear decrease of the error even when the condition on the drift term ($H_{Langevin}$) is not respected. Also except some more extreme cases (Ornstein-Uhlenbeck process with very stiff coefficients) the condition of one collision per time steps seems generally respected. Another important fact is that it has seemed increasingly difficult to obtain a good estimation for the Richardson-Romberg estimator.

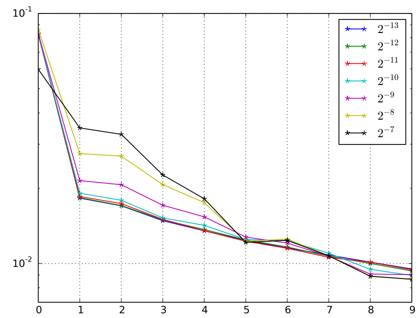
Interesting extensions would be the implementation of a multi-dimensional algorithm and the inclusion in the drift term of actual fluid flow calculations obtained from a DNS for example, which would be more in line with the general goal of the thesis, the simulation of colloidal particles in turbulent flow. Also a better understanding of the various penalisation schemes would be very useful.



(a) $x_0 \;=\; 0.1,\; u_0 = -1.1$

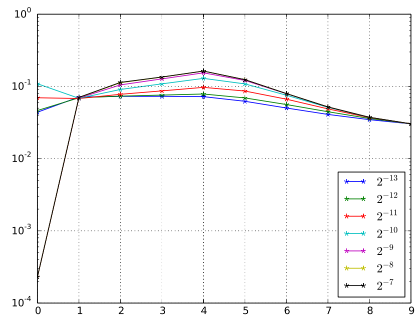


(b) $x_0 \;=\; 0.01,\; u_0 = -0.11$

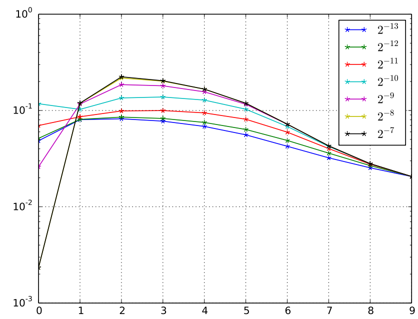


(c) $x_0 \;=\; 0.001,\; u_0 = -0.011$

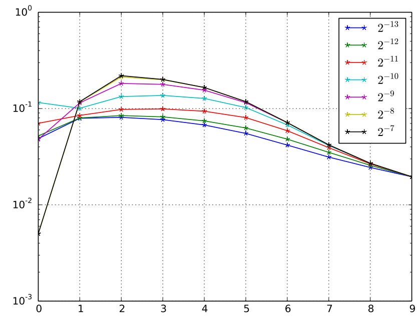
Figure 2.31: Variances $b \equiv \sin$



(a) $x_0 \;=\; 0.1,\; u_0 = -1.1$



(b) $x_0 \;=\; 0.01,\; u_0 = -0.11$



(c) $x_0 \;=\; 0.001,\; u_0 = -0.011$

Figure 2.32: Variances Ornstein-Uhlenbeck case

Appendices

1 Various results regarding the weak error

1.1 $b \equiv 0$

In table (12), we present the results for the estimator $\frac{1}{N_{\text{MC}}}\sum_{n=1}^{N_{\text{MC}}} f(\bar{X}_T^{c,n,i}, \bar{U}_T^{c,n,i})$ for one value of $i \in \{1,\ldots,N_{\text{Err}}\}$

	$x_0 = 0.1, u_0 = -1.1$			$x_0 = 0.01, u_0 = -0.11$			$x_0 = 0.001, u_0 = -0.011$		
Δt	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int
2^{-12}	3.543	13.79	$7.43\cdot10^{-4}$	1.355	3.33	$3.65\cdot10^{-4}$	1.334	3.22	$3.59\cdot10^{-4}$
2^{-11}	3.543	13.79	$7.43\cdot10^{-4}$	1.355	3.33	$3.65\cdot10^{-4}$	1.333	3.22	$3.59\cdot10^{-4}$
2^{-10}	3.543	13.78	$7.43\cdot10^{-4}$	1.355	3.32	$3.65\cdot10^{-4}$	1.333	3.22	$3.59\cdot10^{-4}$
2^{-9}	3.542	13.77	$7.42\cdot10^{-4}$	1.354	3.32	$3.65\cdot10^{-4}$	1.333	3.22	$3.59\cdot10^{-4}$
2^{-8}	3.542	13.76	$7.42\cdot10^{-4}$	1.353	3.32	$3.64\cdot10^{-4}$	1.332	3.21	$3.59\cdot10^{-4}$
2^{-7}	3.540	13.72	$7.41\cdot10^{-4}$	1.351	3.31	$3.64\cdot10^{-4}$	1.330	3.20	$3.58\cdot10^{-4}$
2^{-6}	3.536	13.65	$7.39\cdot10^{-4}$	1.347	3.28	$3.62\cdot10^{-4}$	1.326	3.18	$3.57\cdot10^{-4}$
2^{-5}	3.528	13.51	$7.35\cdot10^{-4}$	1.340	3.24	$3.60\cdot10^{-4}$	1.318	3.14	$3.55\cdot10^{-4}$

Table 12: Results for one estimator in the case $b \equiv 0$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Hits	$\bar{K}_T$	Hits	$\bar{K}_T$	Hits	$\bar{K}_T$
2^{-12}	$1.59\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.45\cdot10^{-5}$	$7.63\cdot10^{-5}$	$4.41\cdot10^{-5}$
2^{-11}	$1.59\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.45\cdot10^{-5}$	$7.63\cdot10^{-5}$	$4.41\cdot10^{-5}$
2^{-10}	$1.59\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.45\cdot10^{-5}$	$7.62\cdot10^{-5}$	$4.41\cdot10^{-5}$
2^{-9}	$1.58\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.45\cdot10^{-5}$	$7.61\cdot10^{-5}$	$4.41\cdot10^{-5}$
2^{-8}	$1.58\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.46\cdot10^{-5}$	$7.59\cdot10^{-5}$	$4.42\cdot10^{-5}$
2^{-7}	$1.56\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.07\cdot10^{-5}$	$4.46\cdot10^{-5}$	$7.53\cdot10^{-5}$	$4.42\cdot10^{-5}$
2^{-6}	$1.54\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.06\cdot10^{-5}$	$4.48\cdot10^{-5}$	$7.38\cdot10^{-5}$	$4.44\cdot10^{-5}$
2^{-5}	$1.48\cdot10^{-5}$	$3.98\cdot10^{-5}$	$6.03\cdot10^{-5}$	$4.51\cdot10^{-5}$	$6.88\cdot10^{-5}$	$4.47\cdot10^{-5}$

Table 13: $1/2-$Confidence Interval $b \equiv 0$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-12}	$4.35{\cdot}10^{-4}$	$2.45{\cdot}10^{-4}$	$1.40{\cdot}10^{-4}$	$8.38{\cdot}10^{-5}$	$2.17{\cdot}10^{-4}$	$8.92{\cdot}10^{-5}$
2^{-11}	$4.34{\cdot}10^{-4}$	$2.74{\cdot}10^{-4}$	$1.45{\cdot}10^{-4}$	$8.65{\cdot}10^{-5}$	$3.10{\cdot}10^{-4}$	$1.16{\cdot}10^{-4}$
2^{-10}	$4.73{\cdot}10^{-4}$	$3.63{\cdot}10^{-4}$	$3.35{\cdot}10^{-4}$	$1.24{\cdot}10^{-4}$	$5.30{\cdot}10^{-4}$	$1.43{\cdot}10^{-4}$
2^{-9}	$8.83{\cdot}10^{-4}$	$4.03{\cdot}10^{-4}$	$8.06{\cdot}10^{-4}$	$1.21{\cdot}10^{-4}$	$9.90{\cdot}10^{-4}$	$1.37{\cdot}10^{-4}$
2^{-8}	$1.86{\cdot}10^{-3}$	$4.00{\cdot}10^{-4}$	$1.78{\cdot}10^{-3}$	$1.26{\cdot}10^{-4}$	$1.97{\cdot}10^{-3}$	$1.33{\cdot}10^{-4}$
2^{-7}	$3.83{\cdot}10^{-3}$	$4.06{\cdot}10^{-4}$	$3.75{\cdot}10^{-3}$	$1.18{\cdot}10^{-4}$	$3.93{\cdot}10^{-3}$	$1.31{\cdot}10^{-4}$
2^{-6}	$7.69{\cdot}10^{-3}$	$4.04{\cdot}10^{-4}$	$7.61{\cdot}10^{-3}$	$1.42{\cdot}10^{-4}$	$7.77{\cdot}10^{-3}$	$1.30{\cdot}10^{-4}$
2^{-5}	$1.54{\cdot}10^{-2}$	$4.21{\cdot}10^{-4}$	$1.53{\cdot}10^{-2}$	$1.16{\cdot}10^{-4}$	$1.55{\cdot}10^{-2}$	$1.26{\cdot}10^{-4}$

Table 14: Result of $\overline{\overline{\text{Error}}}[f]$ in the case $b \equiv 0$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-5}	$2.94{\cdot}10^{-4}$	$1.58{\cdot}10^{-4}$	$1.53{\cdot}10^{-4}$	$6.89{\cdot}10^{-5}$	$1.45{\cdot}10^{-4}$	$7.10{\cdot}10^{-5}$
2^{-4}	$2.99{\cdot}10^{-4}$	$9.56{\cdot}10^{-5}$	$3.26{\cdot}10^{-4}$	$1.24{\cdot}10^{-4}$	$3.61{\cdot}10^{-4}$	$1.26{\cdot}10^{-4}$
2^{-3}	$1.10{\cdot}10^{-3}$	$2.58{\cdot}10^{-4}$	$1.36{\cdot}10^{-3}$	$1.43{\cdot}10^{-4}$	$1.35{\cdot}10^{-3}$	$1.63{\cdot}10^{-4}$
2^{-2}	$5.06{\cdot}10^{-3}$	$2.22{\cdot}10^{-4}$	$5.13{\cdot}10^{-3}$	$1.27{\cdot}10^{-4}$	$5.24{\cdot}10^{-3}$	$1.79{\cdot}10^{-4}$

Table 15: Result of linear estimation Romberg $b = 0$

	$x_0 = 0.1, u_0 = -1.1$			$x_0 = 0.01, u_0 = -0.11$			$x_0 = 0.001, u_0 = -0.011$		
Δt	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int
2^{-12}	0.385	0.112	$6.71{\cdot}10^{-5}$	0.597	0.083	$5.76{\cdot}10^{-5}$	0.599	0.082	$5.73{\cdot}10^{-5}$
2^{-11}	0.385	0.112	$6.71{\cdot}10^{-5}$	0.597	0.083	$5.76{\cdot}10^{-5}$	0.599	0.082	$5.73{\cdot}10^{-5}$
2^{-10}	0.385	0.112	$6.71{\cdot}10^{-5}$	0.597	0.083	$5.76{\cdot}10^{-5}$	0.599	0.082	$5.73{\cdot}10^{-5}$
2^{-9}	0.385	0.112	$6.71{\cdot}10^{-5}$	0.597	0.083	$5.76{\cdot}10^{-5}$	0.599	0.082	$5.73{\cdot}10^{-5}$
2^{-8}	0.384	0.112	$6.71{\cdot}10^{-5}$	0.597	0.083	$5.76{\cdot}10^{-5}$	0.599	0.082	$5.73{\cdot}10^{-5}$
2^{-7}	0.383	0.112	$6.71{\cdot}10^{-5}$	0.596	0.083	$5.77{\cdot}10^{-5}$	0.599	0.082	$5.74{\cdot}10^{-5}$
2^{-6}	0.381	0.113	$6.71{\cdot}10^{-5}$	0.595	0.084	$5.79{\cdot}10^{-5}$	0.598	0.083	$5.76{\cdot}10^{-5}$
2^{-5}	0.377	0.113	$6.71{\cdot}10^{-5}$	0.594	0.085	$5.82{\cdot}10^{-5}$	0.596	0.084	$5.79{\cdot}10^{-5}$

Table 16: Result of one estimator in the case $b \equiv \sin$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-12}	$8.70 \cdot 10^{-5}$	$3.63 \cdot 10^{-5}$	$2.48 \cdot 10^{-5}$	$1.48 \cdot 10^{-5}$	$3.05 \cdot 10^{-5}$	$1.21 \cdot 10^{-5}$
2^{-11}	$1.55 \cdot 10^{-4}$	$3.58 \cdot 10^{-5}$	$1.77 \cdot 10^{-5}$	$8.60 \cdot 10^{-6}$	$3.43 \cdot 10^{-5}$	$1.48 \cdot 10^{-5}$
2^{-10}	$2.93 \cdot 10^{-4}$	$3.36 \cdot 10^{-5}$	$5.60 \cdot 10^{-5}$	$1.58 \cdot 10^{-5}$	$7.31 \cdot 10^{-5}$	$2.27 \cdot 10^{-5}$
2^{-9}	$5.66 \cdot 10^{-4}$	$3.32 \cdot 10^{-5}$	$1.55 \cdot 10^{-4}$	$1.44 \cdot 10^{-5}$	$1.68 \cdot 10^{-4}$	$2.26 \cdot 10^{-5}$
2^{-8}	$1.11 \cdot 10^{-3}$	$3.45 \cdot 10^{-5}$	$3.60 \cdot 10^{-4}$	$2.12 \cdot 10^{-5}$	$3.59 \cdot 10^{-4}$	$2.37 \cdot 10^{-5}$
2^{-7}	$2.20 \cdot 10^{-3}$	$3.58 \cdot 10^{-5}$	$7.70 \cdot 10^{-4}$	$1.63 \cdot 10^{-5}$	$7.39 \cdot 10^{-4}$	$2.20 \cdot 10^{-5}$
2^{-6}	$4.35 \cdot 10^{-3}$	$3.33 \cdot 10^{-5}$	$1.60 \cdot 10^{-3}$	$1.83 \cdot 10^{-5}$	$1.50 \cdot 10^{-3}$	$2.11 \cdot 10^{-5}$
2^{-5}	$8.52 \cdot 10^{-3}$	$2.40 \cdot 10^{-5}$	$3.22 \cdot 10^{-3}$	$2.05 \cdot 10^{-5}$	$3.03 \cdot 10^{-3}$	$2.15 \cdot 10^{-5}$

Table 17: Result of error estimations $b \equiv \sin$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-8}	$1.60 \cdot 10^{-4}$	$2.04 \cdot 10^{-5}$	$2.28 \cdot 10^{-5}$	$1.88 \cdot 10^{-5}$	$5.26 \cdot 10^{-5}$	$3.46 \cdot 10^{-5}$
2^{-7}	$1.47 \cdot 10^{-3}$	$2.79 \cdot 10^{-5}$	$7.65 \cdot 10^{-5}$	$8.42 \cdot 10^{-5}$	$3.76 \cdot 10^{-5}$	$1.69 \cdot 10^{-5}$
2^{-6}	$3.20 \cdot 10^{-3}$	$2.65 \cdot 10^{-5}$	$5.55 \cdot 10^{-5}$	$5.33 \cdot 10^{-5}$	$9.25 \cdot 10^{-5}$	$3.59 \cdot 10^{-5}$
2^{-5}	$2.23 \cdot 10^{-2}$	$3.12 \cdot 10^{-5}$	$4.20 \cdot 10^{-3}$	$9.06 \cdot 10^{-5}$	$1.55 \cdot 10^{-3}$	$4.44 \cdot 10^{-5}$

Table 18: Result Romberg error estimation $b \equiv \sin$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-12}	$2.82 \cdot 10^{-3}$	$1.00 \cdot 10^{-5}$	$3.98 \cdot 10^{-3}$	$3.79 \cdot 10^{-6}$	$4.11 \cdot 10^{-3}$	$6.02 \cdot 10^{-6}$
2^{-11}	$4.83 \cdot 10^{-3}$	$1.03 \cdot 10^{-5}$	$4.42 \cdot 10^{-3}$	$6.75 \cdot 10^{-6}$	$4.37 \cdot 10^{-3}$	$6.65 \cdot 10^{-6}$
2^{-10}	$1.95 \cdot 10^{-2}$	$7.95 \cdot 10^{-6}$	$2.05 \cdot 10^{-2}$	$8.90 \cdot 10^{-6}$	$2.06 \cdot 10^{-2}$	$1.09 \cdot 10^{-5}$
2^{-9}	$4.68 \cdot 10^{-2}$	$1.28 \cdot 10^{-5}$	$5.03 \cdot 10^{-2}$	$1.05 \cdot 10^{-5}$	$5.06 \cdot 10^{-2}$	$1.09 \cdot 10^{-5}$
2^{-8}	$9.41 \cdot 10^{-2}$	$1.01 \cdot 10^{-5}$	$1.02 \cdot 10^{-1}$	$1.46 \cdot 10^{-5}$	$1.02 \cdot 10^{-1}$	$1.17 \cdot 10^{-5}$
2^{-7}	$1.67 \cdot 10^{-1}$	$9.85 \cdot 10^{-6}$	$1.81 \cdot 10^{-1}$	$1.68 \cdot 10^{-5}$	$1.82 \cdot 10^{-1}$	$2.05 \cdot 10^{-5}$
2^{-6}	$2.64 \cdot 10^{-1}$	$1.98 \cdot 10^{-5}$	$2.83 \cdot 10^{-1}$	$2.31 \cdot 10^{-5}$	$2.85 \cdot 10^{-1}$	$2.57 \cdot 10^{-5}$
2^{-5}	$3.95 \cdot 10^{-1}$	$2.48 \cdot 10^{-5}$	$3.78 \cdot 10^{-1}$	$2.54 \cdot 10^{-5}$	$3.77 \cdot 10^{-1}$	$2.03 \cdot 10^{-5}$

Table 19: Result of Error Estimation in the Ornstein Uhlenbeck case

	$x_0 = 0.1, u_0 = -1.1$			$x_0 = 0.01, u_0 = -0.11$			$x_0 = 0.001, u_0 = -0.011$		
Δt	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int	Result	Var	$1/2-$Conf Int
2^{-12}	0.672	0.015	$2.48 \cdot 10^{-5}$	0.704	0.009	$1.87 \cdot 10^{-5}$	0.707	0.008	$1.81 \cdot 10^{-5}$
2^{-11}	0.664	0.017	$2.58 \cdot 10^{-5}$	0.695	0.010	$2.01 \cdot 10^{-5}$	0.698	0.009	$1.95 \cdot 10^{-5}$
2^{-10}	0.650	0.019	$2.78 \cdot 10^{-5}$	0.679	0.013	$2.26 \cdot 10^{-5}$	0.682	0.012	$2.20 \cdot 10^{-5}$
2^{-9}	0.622	0.025	$3.14 \cdot 10^{-5}$	0.649	0.018	$2.70 \cdot 10^{-5}$	0.652	0.018	$2.65 \cdot 10^{-5}$
2^{-8}	0.575	0.035	$3.74 \cdot 10^{-5}$	0.598	0.029	$3.41 \cdot 10^{-5}$	0.600	0.029	$3.38 \cdot 10^{-5}$
2^{-7}	0.502	0.052	$4.58 \cdot 10^{-5}$	0.518	0.048	$4.38 \cdot 10^{-5}$	0.520	0.048	$4.36 \cdot 10^{-5}$
2^{-6}	0.405	0.075	$5.46 \cdot 10^{-5}$	0.416	0.072	$5.38 \cdot 10^{-5}$	0.418	0.072	$5.37 \cdot 10^{-5}$
2^{-5}	0.274	0.086	$5.88 \cdot 10^{-5}$	0.322	0.092	$6.06 \cdot 10^{-5}$	0.326	0.091	$6.04 \cdot 10^{-5}$

Table 20: Result of one estimator in the Ornstein-Uhlenbeck

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-11}	$1.64 \cdot 10^{-5}$	$8.44 \cdot 10^{-6}$	$8.70 \cdot 10^{-6}$	$6.02 \cdot 10^{-6}$	$1.78 \cdot 10^{-5}$	$6.65 \cdot 10^{-6}$
2^{-10}	$1.47 \cdot 10^{-2}$	$7.95 \cdot 10^{-6}$	$1.61 \cdot 10^{-2}$	$8.90 \cdot 10^{-6}$	$1.63 \cdot 10^{-2}$	$1.09 \cdot 10^{-5}$
2^{-9}	$4.19 \cdot 10^{-2}$	$1.28 \cdot 10^{-5}$	$4.59 \cdot 10^{-2}$	$1.05 \cdot 10^{-5}$	$4.62 \cdot 10^{-2}$	$1.09 \cdot 10^{-5}$
2^{-8}	$8.92 \cdot 10^{-2}$	$1.01 \cdot 10^{-5}$	$9.74 \cdot 10^{-2}$	$1.46 \cdot 10^{-5}$	$9.81 \cdot 10^{-2}$	$1.17 \cdot 10^{-5}$
2^{-7}	$1.63 \cdot 10^{-1}$	$9.85 \cdot 10^{-6}$	$1.77 \cdot 10^{-1}$	$1.68 \cdot 10^{-5}$	$1.78 \cdot 10^{-1}$	$2.05 \cdot 10^{-5}$
2^{-6}	$2.59 \cdot 10^{-1}$	$1.98 \cdot 10^{-5}$	$2.79 \cdot 10^{-1}$	$2.31 \cdot 10^{-5}$	$2.81 \cdot 10^{-1}$	$2.57 \cdot 10^{-5}$
2^{-5}	$3.90 \cdot 10^{-1}$	$2.48 \cdot 10^{-5}$	$3.73 \cdot 10^{-1}$	$2.54 \cdot 10^{-5}$	$3.73 \cdot 10^{-1}$	$2.03 \cdot 10^{-5}$

Table 21: Result of error estimation in the Ornstein-Uhlenbeck case - MC reference

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-11}	$1.05 \cdot 10^{-2}$	$1.81 \cdot 10^{-5}$	$1.24 \cdot 10^{-2}$	$8.90 \cdot 10^{-6}$	$1.26 \cdot 10^{-2}$	$1.41 \cdot 10^{-5}$
2^{-10}	$9.86 \cdot 10^{-3}$	$1.86 \cdot 10^{-5}$	$1.17 \cdot 10^{-2}$	$1.63 \cdot 10^{-5}$	$1.19 \cdot 10^{-2}$	$1.64 \cdot 10^{-5}$
2^{-9}	$7.71 \cdot 10^{-3}$	$1.63 \cdot 10^{-5}$	$9.25 \cdot 10^{-3}$	$2.11 \cdot 10^{-5}$	$9.37 \cdot 10^{-3}$	$1.73 \cdot 10^{-5}$
2^{-8}	$5.37 \cdot 10^{-4}$	$3.21 \cdot 10^{-5}$	$1.20 \cdot 10^{-3}$	$2.98 \cdot 10^{-5}$	$1.27 \cdot 10^{-3}$	$2.12 \cdot 10^{-5}$
2^{-7}	$2.07 \cdot 10^{-2}$	$2.59 \cdot 10^{-5}$	$2.24 \cdot 10^{-2}$	$2.76 \cdot 10^{-5}$	$2.26 \cdot 10^{-2}$	$3.71 \cdot 10^{-5}$
2^{-6}	$7.05 \cdot 10^{-2}$	$1.40 \cdot 10^{-5}$	$7.92 \cdot 10^{-2}$	$4.10 \cdot 10^{-5}$	$7.98 \cdot 10^{-2}$	$4.49 \cdot 10^{-5}$
2^{-5}	$1.33 \cdot 10^{-1}$	$3.52 \cdot 10^{-5}$	$1.89 \cdot 10^{-1}$	$4.67 \cdot 10^{-5}$	1.93	$5.99 \cdot 10^{-5}$

Table 22: Result of Romberg error estimation in the Ornstein-Uhlenbeck case

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-11}	$9.98{\cdot}10^{-3}$	$1.02{\cdot}10^{-6}$	$2.96{\cdot}10^{-2}$	$3.34{\cdot}10^{-6}$	$4.37{\cdot}10^{-2}$	$4.41{\cdot}10^{-6}$
2^{-10}	$1.55{\cdot}10^{-2}$	$1.34{\cdot}10^{-6}$	$3.77{\cdot}10^{-2}$	$3.77{\cdot}10^{-6}$	$5.27{\cdot}10^{-2}$	$4.82{\cdot}10^{-6}$
2^{-9}	$2.23{\cdot}10^{-2}$	$1.72{\cdot}10^{-6}$	$4.68{\cdot}10^{-2}$	$4.21{\cdot}10^{-6}$	$6.20{\cdot}10^{-2}$	$5.19{\cdot}10^{-6}$
2^{-8}	$3.16{\cdot}10^{-2}$	$2.21{\cdot}10^{-6}$	$5.80{\cdot}10^{-2}$	$4.69{\cdot}10^{-6}$	$7.25{\cdot}10^{-2}$	$5.56{\cdot}10^{-6}$
2^{-7}	$4.43{\cdot}10^{-2}$	$2.87{\cdot}10^{-6}$	$7.21{\cdot}10^{-2}$	$5.28{\cdot}10^{-6}$	$8.50{\cdot}10^{-2}$	$5.98{\cdot}10^{-6}$
2^{-6}	$6.20{\cdot}10^{-2}$	$3.77{\cdot}10^{-6}$	$9.02{\cdot}10^{-2}$	$6.01{\cdot}10^{-6}$	$1.01{\cdot}10^{-1}$	$6.49{\cdot}10^{-6}$
2^{-5}	$8.71{\cdot}10^{-2}$	$5.00{\cdot}10^{-6}$	$1.14{\cdot}10^{-1}$	$6.99{\cdot}10^{-6}$	$1.21{\cdot}10^{-1}$	$7.15{\cdot}10^{-6}$

Table 23: Result of strong error estimation $b \equiv 0$ for $\bar{X}$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-11}	$1.71{\cdot}10^{-2}$	$4.34{\cdot}10^{-6}$	$6.83{\cdot}10^{-2}$	$1.04{\cdot}10^{-5}$	$9.33{\cdot}10^{-2}$	$1.23{\cdot}10^{-5}$
2^{-10}	$2.49{\cdot}10^{-2}$	$5.28{\cdot}10^{-6}$	$8.30{\cdot}10^{-2}$	$1.15{\cdot}10^{-5}$	$1.09{\cdot}10^{-1}$	$1.33{\cdot}10^{-5}$
2^{-9}	$3.43{\cdot}10^{-2}$	$6.21{\cdot}10^{-6}$	$9.90{\cdot}10^{-2}$	$1.26{\cdot}10^{-5}$	$1.25{\cdot}10^{-1}$	$1.41{\cdot}10^{-5}$
2^{-8}	$4.65{\cdot}10^{-2}$	$7.22{\cdot}10^{-6}$	$1.18{\cdot}10^{-1}$	$1.37{\cdot}10^{-5}$	$1.42{\cdot}10^{-1}$	$1.50{\cdot}10^{-5}$
2^{-7}	$6.27{\cdot}10^{-2}$	$8.36{\cdot}10^{-6}$	$1.40{\cdot}10^{-1}$	$1.48{\cdot}10^{-5}$	$1.62{\cdot}10^{-1}$	$1.58{\cdot}10^{-5}$
2^{-6}	$8.42{\cdot}10^{-2}$	$9.67{\cdot}10^{-6}$	$1.67{\cdot}10^{-1}$	$1.60{\cdot}10^{-5}$	$1.85{\cdot}10^{-1}$	$1.67{\cdot}10^{-5}$
2^{-5}	$1.13{\cdot}10^{-1}$	$1.12{\cdot}10^{-5}$	$2.00{\cdot}10^{-1}$	$1.73{\cdot}10^{-5}$	$2.11{\cdot}10^{-1}$	$1.77{\cdot}10^{-5}$

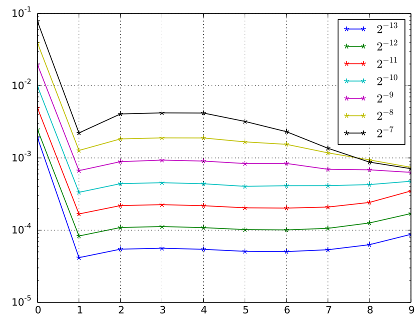
Table 24: Result of strong error estimation $b \equiv 0$ for $\bar{U}$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int	Result	$1/2-$Conf Int
2^{-11}	$1.06{\cdot}10^{-2}$	$1.02{\cdot}10^{-6}$	$2.30{\cdot}10^{-2}$	$2.68{\cdot}10^{-6}$	$3.30{\cdot}10^{-2}$	$3.54{\cdot}10^{-6}$
2^{-10}	$1.64{\cdot}10^{-2}$	$1.37{\cdot}10^{-6}$	$2.93{\cdot}10^{-2}$	$3.05{\cdot}10^{-6}$	$3.98{\cdot}10^{-2}$	$3.88{\cdot}10^{-6}$
2^{-9}	$2.37{\cdot}10^{-2}$	$1.80{\cdot}10^{-6}$	$3.63{\cdot}10^{-2}$	$3.43{\cdot}10^{-6}$	$4.70{\cdot}10^{-2}$	$4.20{\cdot}10^{-6}$
2^{-8}	$3.36{\cdot}10^{-2}$	$2.37{\cdot}10^{-6}$	$4.51{\cdot}10^{-2}$	$3.86{\cdot}10^{-6}$	$5.52{\cdot}10^{-2}$	$4.53{\cdot}10^{-6}$
2^{-7}	$4.72{\cdot}10^{-2}$	$3.14{\cdot}10^{-6}$	$5.63{\cdot}10^{-2}$	$4.38{\cdot}10^{-6}$	$6.51{\cdot}10^{-2}$	$4.90{\cdot}10^{-6}$
2^{-6}	$6.63{\cdot}10^{-2}$	$4.17{\cdot}10^{-6}$	$7.13{\cdot}10^{-2}$	$5.02{\cdot}10^{-6}$	$7.80{\cdot}10^{-2}$	$5.33{\cdot}10^{-6}$
2^{-5}	$9.36{\cdot}10^{-2}$	$5.46{\cdot}10^{-6}$	$9.18{\cdot}10^{-2}$	$5.77{\cdot}10^{-6}$	$9.54{\cdot}10^{-2}$	$5.83{\cdot}10^{-6}$

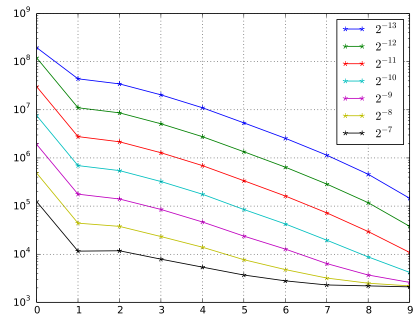
Table 25: Result of strong error estimation $b \equiv \sin$ for $\bar{X}$

	$x_0 = 0.1, u_0 = -1.1$		$x_0 = 0.01, u_0 = -0.11$		$x_0 = 0.001, u_0 = -0.011$	
Δt	Result	1/2−Conf Int	Result	1/2−Conf Int	Result	1/2−Conf Int
2^{-11}	$2.14{\cdot}10^{-2}$	$5.37{\cdot}10^{-6}$	$6.41{\cdot}10^{-2}$	$1.11{\cdot}10^{-5}$	$8.71{\cdot}10^{-2}$	$1.35{\cdot}10^{-5}$
2^{-10}	$3.11{\cdot}10^{-2}$	$6.64{\cdot}10^{-6}$	$7.93{\cdot}10^{-2}$	$1.25{\cdot}10^{-5}$	$1.03{\cdot}10^{-1}$	$1.47{\cdot}10^{-5}$
2^{-9}	$4.30{\cdot}10^{-2}$	$7.97{\cdot}10^{-6}$	$9.62{\cdot}10^{-2}$	$1.39{\cdot}10^{-5}$	$1.20{\cdot}10^{-1}$	$1.59{\cdot}10^{-5}$
2^{-8}	$5.85{\cdot}10^{-2}$	$9.48{\cdot}10^{-6}$	$1.16{\cdot}10^{-1}$	$1.54{\cdot}10^{-5}$	$1.39{\cdot}10^{-1}$	$1.71{\cdot}10^{-5}$
2^{-7}	$7.95{\cdot}10^{-2}$	$1.13{\cdot}10^{-5}$	$1.42{\cdot}10^{-1}$	$1.72{\cdot}10^{-5}$	$1.62{\cdot}10^{-1}$	$1.85{\cdot}10^{-5}$
2^{-6}	$1.08{\cdot}10^{-1}$	$1.35{\cdot}10^{-5}$	$1.75{\cdot}10^{-1}$	$1.93{\cdot}10^{-5}$	$1.91{\cdot}10^{-1}$	$2.01{\cdot}10^{-5}$
2^{-5}	$1.49{\cdot}10^{-1}$	$1.61{\cdot}10^{-5}$	$2.21{\cdot}10^{-1}$	$2.17{\cdot}10^{-5}$	$2.30{\cdot}10^{-1}$	$2.20{\cdot}10^{-5}$

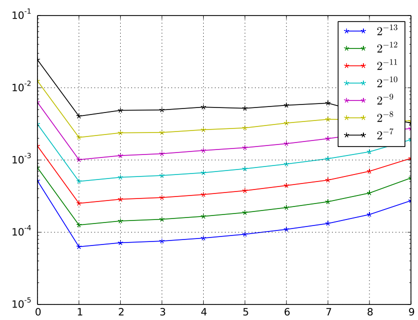
Table 26: Result of strong error estimation $b \equiv \sin$ for $\bar{U}$



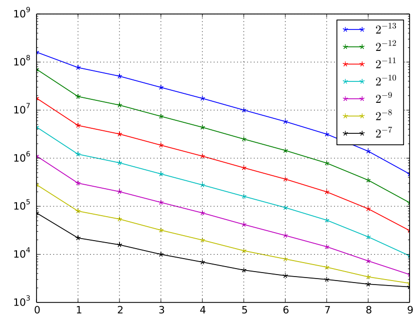
(a) 1/2−CI at 95%, $x_0 = 0.1,\ u_0 = -1.1$.



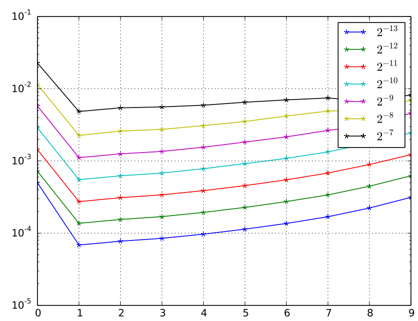
(b) Trajectories $x_0 = 0.1,\ u_0 = -1.1$.



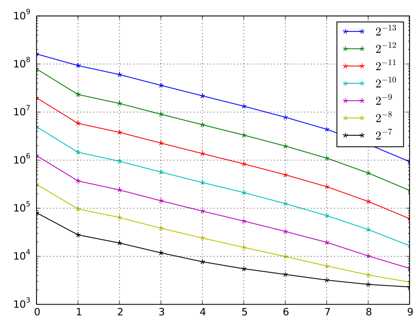
(c) 1/2−CI at 95%, , $x_0 = 0.01,\ u_0 = -0.11$.



(d) Trajectories $x_0 = 0.01,\ u_0 = -0.11$.

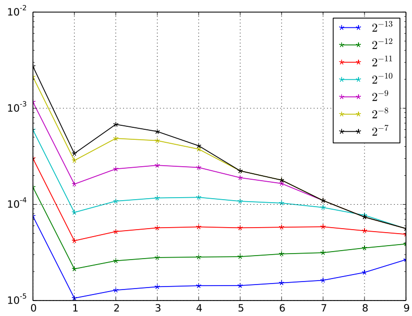


(e) 1/2−CI at 95%, , $x_0 = 0.001,\ u_0 = -0.011$.

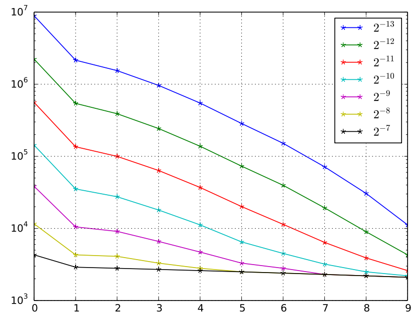


(f) Trajectories $x_0 = 0.001,\ u_0 = -0.011$.

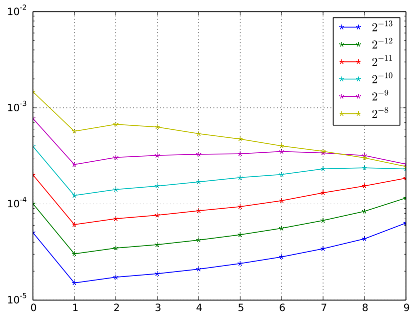
Figure 33: Confidence Interval size and Number of trajectories $b \equiv 0$



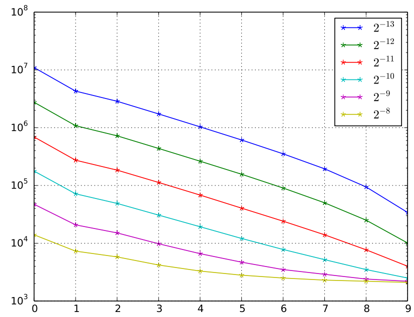
(a) $1/2-$CI at 95%, $x_0 = 0.1,\ u_0 = -1.1$.



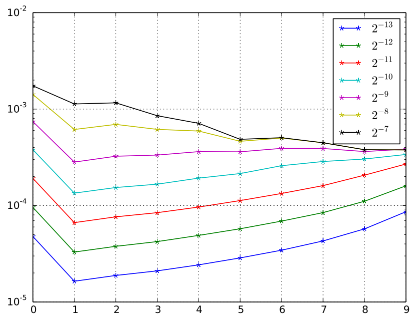
(b) Trajectories $x_0 = 0.1,\ u_0 = -1.1$.



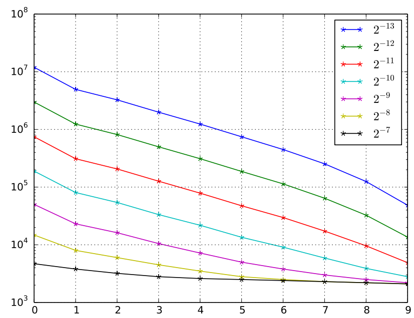
(c) $1/2-$CI at 95%, , $x_0 = 0.01,\ u_0 = -0.11$.



(d) Trajectories $x_0 = 0.01,\ u_0 = -0.11$.

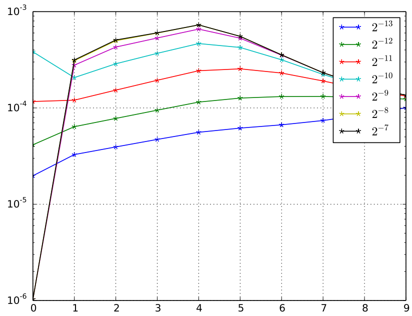


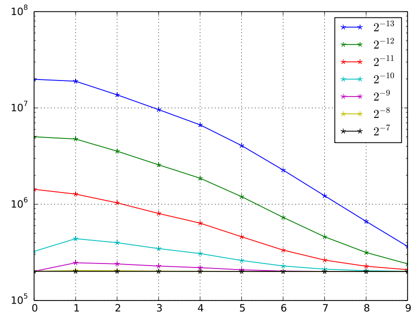
(e) $1/2-$CI at 95%, , $x_0 = 0.001,\ u_0 = -0.011$.



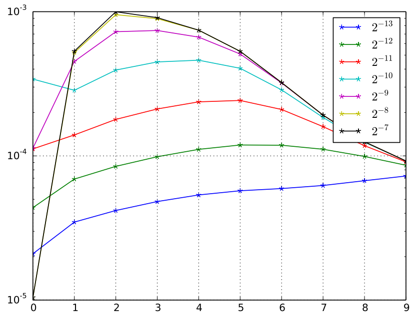
(f) Trajectories $x_0 = 0.001,\ u_0 = -0.011$.

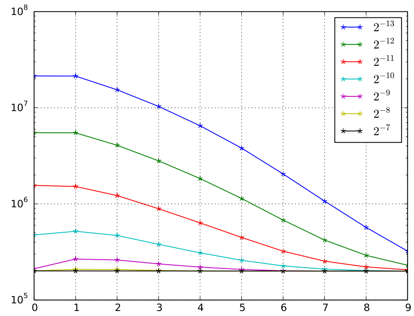
Figure 34: Confidence Interval size and Number of trajectories $b \equiv \sin$



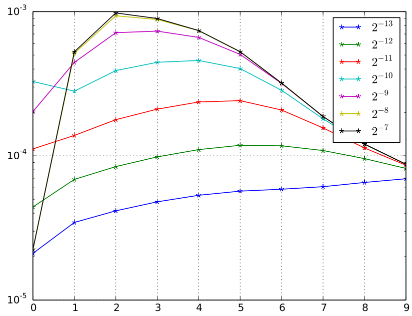


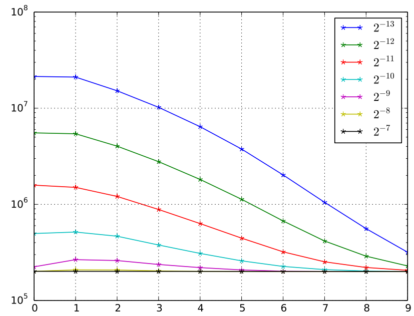
(a) $1/2-$CI at 95%, $x_0 = 0.1,\ u_0 = -1.1.$ (b) Trajectories $x_0 = 0.1,\ u_0 = -1.1.$





(c) $1/2-$CI at 95%, , $x_0 = 0.01,\ u_0 = -0.11.$ (d) Trajectories $x_0 = 0.01,\ u_0 = -0.11.$





(e) $1/2-$CI at 95%, , $x_0 = 0.001,\ u_0 = -0.011.$ (f) Trajectories $x_0 = 0.001,\ u_0 = -0.011.$

Figure 35: Confidence Interval size and Number of trajectories Ornstein-Uhlenbeck

Chapter 3

Non-asymptotic Approximations of the Langevin Equation by a Diffusion in the case of particle collision

1 Introduction

The two previous chapters were focused on the analysis of discretisation schemes of Langevin models with specular reflection boundary conditions. We shall consider a different aspect in this chapter that was mentioned in the Introduction. It is known that there are several different convergence possibilities when the diffusion coefficient or drift terms of the components in a kinetic model go to infinity or 0. Historically, the convergence of the Langevin model towards the Einstein Brownian model for particles has been called the over-damped Langevin limit. Such a limit is taken when assuming the process is ergodic, but in our case, due to the presence of turbulence, for example the process is not at equilibrium. We shall consider the SDE

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s\,ds \\ u_t = u_0 - \beta \displaystyle\int_0^t u_s\,ds + \beta \int_0^t \mu(x_s)\,ds + \beta\sigma W_t \end{cases} \tag{1.1}$$

and compare it, in an non-asymptotic manner, to the process

$$\begin{cases} Y_t = x_0 + \displaystyle\int_0^t \mu(Y_s)\,ds + \sigma W_t\,. \end{cases} \tag{1.2}$$

There are several asymptotic results that exist. In [Pavliotis, 2014], the over-damped Langevin limit towards the Einstein model of Brownian motion is presented. We consider the process:

$$\begin{cases} X_t^\varepsilon = X_0 + \displaystyle\int_0^t U_s^\varepsilon\,ds \\ U_t^\varepsilon = U_0 - \displaystyle\int_0^t \nabla V(X_s^\varepsilon)\,ds - \varepsilon^{-2}\int_0^t U_s^\varepsilon\,ds + \varepsilon^{-1}\sigma W_t \end{cases} \tag{1.3}$$

where V is a potential. By considering the time-change $t \to \varepsilon^{-2}t$ and scaling property of the Brownian motion, then $(X_t^\varepsilon)_{t\geq 0}$ converges to the diffusion process $(X_t)_{t\geq 0}$ such that $X_t = X_0 - \int_0^t \nabla V(X_s)\,ds + \sigma\widetilde{W}_t$ as $\varepsilon \to 0$. This is called an over-damped limit because in the equation (1.3), which is an application of Newton's equations, the term in ε^{-2} corresponds to a drag force and we take the limit in $\varepsilon^{-1} \to +\infty$.

The case of the under-damped limit is more complicated but, in some cases there exists a diffusion equation for the Hamiltonian.

Another type of convergence was presented in [Pardoux and Veretennikov, 2001]. Consider the following SDE:

$$\begin{cases} X_t^\varepsilon = X_0 + \int_0^t F(X^\varepsilon, U_s^\varepsilon)\, ds + \varepsilon^{-1} \int_0^t G(X^\varepsilon, U_s^\varepsilon)\, ds + \int_0^t H(X^\varepsilon, U_s^\varepsilon)\, dB_s^\varepsilon \\ U_t^\varepsilon = U_0 + \varepsilon^{-2} \int_0^t b(U_s^\varepsilon)\, ds + \varepsilon^{-1} \int_0^t \sigma(U_s^\varepsilon)\, dB_s^\varepsilon . \end{cases} \tag{1.4}$$

Provided that the process $(U_t^1)_{t\geq 0}$ has an invariant measure and regularity on the coefficients, the process $(X_t^\varepsilon)_{t\geq 0}$ converges weakly towards a diffusion process. The proof uses corrections based on the solution f of the Poisson equation $\left(b\nabla_u + \frac{\sigma^2}{2}\triangle_u\right) f = -\langle \nabla_u h, G(x,u)\rangle$, where h is a sufficiently smooth test function.

Another type of asymptotics is the Smoluchowski-Kramers limit:

$$\begin{cases} X_t^\varepsilon = X_0 + \int_0^t U_s^\varepsilon\, ds \\ U_t^\varepsilon = U_0 + \varepsilon^{-1} \int_0^t F(X_s^\varepsilon)\, ds - \varepsilon^{-1} \int_0^t U_s^\varepsilon\, ds + \varepsilon^{-1}\sigma W_t \end{cases} \tag{1.5}$$

and it is shown in [Nelson, 1967] or [Karatzas and Shreve, 1991], that for any $T > 0$

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} |X_t^\varepsilon - X_t| = 0 \quad \text{a.s.}$$

where $(X_t)_{t\geq 0}$ verifies

$$X_t = X_0 + \int_0^t F(X_s)\, ds + \sigma W_t\, .$$

Finally, we mention the results in [Spiliopoulos, 2007], which show that the Langevin process with specular boundary conditions converges in a certain sense, in a Smoluchowski-Kramers asymptotic, to a Skorokhod reflected diffusion.

We recall that in this chapter we do not look directly at an asymptotics, but try to calculate the error between (1.6) and the diffusion (1.7). Finally, we compare the same situation, except that we take into account specular reflection on the position in the Langevin case and Skorokhod reflection on the diffusion to which we compare the Langevin.

In order to prove these results, we work with the mild equation.

In the case of the Smoluchowski-Kramers asymptotics, in [Pavliotis, 2014], it is shown that the strong error can be bounded in $\sqrt{\varepsilon}$. In [Hagan *et al.*, 1989], the authors present an approximation of the exit times from a bounded or unbounded domain of the position in the case of a damped-Langevin asymptotics.

1.1 Model

We consider two processes $(x_t, u_t)_{0\leq t\leq T}$ and $(Y_t)_{0\leq t\leq T}$ defined as

$$\begin{cases} x_t = x_0 + \int_0^t u_s\, ds \\ u_t = u_0 - \beta \int_0^t u_s\, ds + \beta \int_0^t \mu(x_s)\, ds + \beta\sigma W_t \end{cases} \tag{1.6}$$

and

$$\left\{ Y_t = x_0 + \int_0^t \mu(Y_s)\, ds + \sigma W_t \,. \right. \tag{1.7}$$

The solution to (1.6) is

$$u_t = \exp(-\beta t)u_0 + \beta \int_0^t \exp(-\beta(t-s))\mu(x_s)\, ds + \beta\sigma \int_0^t \exp(-\beta(t-s))\, dW_s$$

and we have that:

$$x_t = Y_t + \frac{1}{\beta}(u_0 - u_t) + \int_0^t (\mu(x_s) - \mu(Y_s))\, ds \,. \tag{1.8}$$

We mention that from one line of calculation to another, the bounding terms might change value but the notations will remain. Also in subscript, we write according to what parameters those bounding terms depend. Also, for the sake of simplicity, we shall assume $\beta \geq 1$.

1.2 Strong Error

Let μ be a function such that for any $x \in \mathbb{R}$, $|\mu(x)| \leq C_\mu(1+|x|)$, where C_μ does not depend on β. We have that:

$$\mathbb{E}\left[\sup_{s\in[0,t]} |u_s|\right] \leq u_0 + \beta\mathbb{E}\left[\sup_{s\in[0,t]} \int_0^s e^{-\beta(s-r)}|\mu(x_r)|\, dr\right] + \sigma\beta\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s e^{-\beta(s-r)}\, dW_r\right|\right]$$

We consider each term:

$$\begin{aligned}
&\int_0^s e^{-\beta(s-r)}|\mu(x_r)|\, dr \leq C_\mu \int_0^s e^{-\beta(s-r)}(1+|x_r|)\, dr \\
&\leq C_\mu \int_0^s e^{-\beta(s-r)} \left(1 + |x_0| + \left|\int_0^r u_\tau\, d\tau\right|\right) dr \\
&\leq C_\mu(1+|x_0|)(1-e^{-\beta s})\frac{1}{\beta} + C_\mu \int_0^s e^{-\beta(s-r)} \int_0^r |u_\tau|\, d\tau dr \\
&\leq C_\mu(1+|x_0|)\frac{1}{\beta} + \frac{C_\mu}{\beta}\int_0^s (1-e^{-\beta(s-r)})|u_r|\, dr \\
&\leq C_\mu(1+|x_0|)\frac{1}{\beta} + \frac{C_\mu}{\beta}\int_0^s |u_r|\, dr
\end{aligned}$$

meaning that:

$$\begin{aligned}
\mathbb{E}\left[\int_0^s e^{-\beta(s-r)}|\mu(x_r)|\, dr\right] &\leq C_\mu(1+|x_0|)\frac{1}{\beta} + \frac{C_\mu}{\beta}\int_0^t \mathbb{E}|u_r|\, dr \\
&\leq C_\mu(1+|x_0|)\frac{1}{\beta} + \frac{C_\mu}{\beta}\int_0^t \mathbb{E}\left[\sup_{s\in[0,r]} |u_s|\, dr\right] .
\end{aligned}$$

In [Blount and Bose, 2000], the following inequality is shown:

$$\mathbb{P}\left[\sup_{s\in[0,t]} \left|\int_0^s e^{-\beta(s-r)}\, dW_r\right| \geq q\right] \leq 4\beta \exp\left(-\frac{\beta q^2}{4e^{4t}}\right) . \tag{1.9}$$

So, by integrating q between 0 and $+\infty$, we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s e^{-\beta(s-r)}\, dW_r\right|\right] \leq \frac{\sqrt{2\pi}e^{2t}}{\sqrt{\beta}} . \tag{1.10}$$

By combining these various results, we obtain that

$$\mathbb{E}\left[\sup_{s\in[0,t]}|u_s|\right] \leq |u_0| + C_\mu(1+|x_0|) + C_\mu\int_0^t \mathbb{E}\left[\sup_{s\in[0,r]}|u_s|\,dr\right] + \sqrt{2\pi}\sigma e^{2t}\sqrt{\beta}\,. \tag{1.11}$$

So Gronwall's inequality gives that for any $\beta \geq 1$

$$\mathbb{E}\left[\sup_{s\in[0,t]}|u_s|\right] \leq C_{\mu,\sigma,t,x_0,u_0}\sqrt{\beta}\,, \tag{1.12}$$

where

$$C_{\mu,\sigma,t,x_0,u_0} = 2e^{2C_\mu t}\max\left\{|u_0| + C_\mu(1+|x_0|), \sqrt{2\pi}\sigma e^{2t}\right\}.$$

We consider (1.8) and assuming μ is Lipschitz, with Lipschitz constant L_μ uniformly in β:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}|x_s - Y_s|\right] &\leq \frac{1}{\beta}\mathbb{E}\left[\sup_{s\in[0,t]}|u_0 - u_s|\right] + \int_0^t \mathbb{E}\left|\mu(x_s)-\mu(Y_s)\right|\,ds \\ &\leq C_{\mu,\sigma,t,x_0,u_0}\frac{1}{\beta}\sqrt{\beta} + L_\mu\int_0^t \mathbb{E}\left|x_s - Y_s\right|\,ds \\ &\leq C_{\mu,\sigma,t,x_0,u_0}\exp\left(L_\mu t\right)\frac{1}{\sqrt{\beta}}\,,\end{aligned} \tag{1.13}$$

thus concluding that we have a control in $\dfrac{1}{\sqrt{\beta}}$ of the strong error.

2 Weak error

For a function f and initial measure μ_0 that have suitable regularity properties, we try to obtain the following inequality, for any $t > 0$:

$$|\mathbb{E}_{\mu_0} f(x_t) - \mathbb{E}_{\mu_0} f(Y_t)| \leq Cg(\beta)\,,$$

where $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(\beta) \to 0$ as $\beta \to \infty$ and C does not depend on β.

Hypotheses 2.1
We consider the following set of hypotheses ($H_{\text{Weak Bound}}$):

($H_{Backward}$) $f\colon \mathbb{R} \mapsto \mathbb{R}$ is derivable 6 times with bounded derivatives

($H_{Forward}$)-(i) $\mu_0\colon (x,u) \in (\mathbb{R}\times\mathbb{R}) \mapsto [0,1]$ is a probability measure with density that we also denote as μ_0, such that $\partial_x\mu_0, \partial_{xx}\mu_0 \in L^1(\mathbb{R}\times\mathbb{R}) \cap L^\infty(\mathbb{R}\times\mathbb{R})$, the integrals $\int_{\mathbb{R}\times\mathbb{R}}(|u|+u^2)\mu_0(dx,du)$ and $\int_{\mathbb{R}\times\mathbb{R}}(|u|+u^2)|\partial_x\mu_0|(x,u)\,dxdu$ are bounded and μ_0 vanishes at infinity

($H_{Forward}$)-(ii) $\mu\colon \mathbb{R} \mapsto \mathbb{R}$ is bounded and $\mu', \mu'' \in L^\infty(\mathbb{R})$.

The subsections Toy example: Constant drift case and General drift deal with processes that have for domain the entire space while the section Application to reflection deals with reflected processes. The bounds are proven using Taylor's formula, so the main difficulties are to obtain suitable controls of the various moments of the processes.

The subsection Toy example: Constant drift case is based on a backward interpretation, which requires regularity on the test function f , while the subsection General drift utilises a mild equation on the density of the processes, which requires regularity on the initial density μ_0. This is the reason for presenting two sets of hypotheses ($H_{Backward}$) and ($H_{Forward}$).

2.1 Toy example: Constant drift case

In order to better understand what arguments are needed for the general proof, we consider a toy example where the drift $\mu\colon x \mapsto \mu$ is a constant, and $\mu_0 \equiv \delta_{x_0,u_0}$, where δ is the Dirac delta distribution.

Lemma 2.2. *Let $f \in \mathcal{C}_b^6(\mathbb{R})$, then for any $t > 0$ and $(x_0, u_0) \in \mathbb{R} \times \mathbb{R}$, there exists a constant $C_{\mu,\sigma,f,\ldots,f^{(6)},x_0,u_0,t}$, independent of β, such that*

$$|\mathbb{E}f(x_t) - \mathbb{E}f(Y_t)| \leq C_{\mu,\sigma,f,\ldots,f^{(6)},x_0,u_0,t}\frac{1}{\beta}. \tag{2.1}$$

Proof. We apply Ito's formula and by (1.8):

$$\begin{aligned} f(x_t) - f(Y_t) &= \int_0^t f'(x_s)u_s\,ds - \int_0^t f'(Y_s)\,dY_s - \frac{\sigma^2}{2}\int_0^t f''(Y_s)\,ds \\ &= \int_0^t f'\left(Y_s + \frac{1}{\beta}(u_0 - u_s)\right)u_s\,ds - \int_0^t f'(Y_s)\,dY_s - \frac{\sigma^2}{2}\int_0^t f''(Y_s)\,ds\,. \end{aligned} \tag{2.2}$$

We consider the first term under the integral, for any $\omega \in \Omega$:

$$\begin{aligned} f'\left(Y_s(\omega) + \frac{1}{\beta}(u_0 - u_s(\omega))\right) &= f'(Y_s(\omega)) + \left(\frac{1}{\beta}(u_0 - u_s(\omega))\right)f''(Y_s(\omega)) \\ &+ \frac{1}{2}\left(\frac{1}{\beta}(u_0 - u_s(\omega))\right)^2 f^{(3)}(Y_s(\omega)) \\ &+ \frac{1}{2}\left(\frac{1}{\beta}(u_0 - u_s(\omega))\right)^3 \int_0^1 (1-\theta)^2 f^{(4)}\left(Y_s(\omega) + \theta\left(\frac{1}{\beta}(u_0 - u_s(\omega))\right)\right)d\theta\,. \end{aligned} \tag{2.3}$$

Going back to (2.2) and taking the expectation, we obtain that

$$\begin{aligned} \mathbb{E}f(x_t) - \mathbb{E}f(Y_t) &= \mathbb{E}\left[\int_0^t f'(Y_s)u_s\,ds\right] + \mathbb{E}\left[\int_0^t \left(\frac{u_0 - u_s}{\beta}\right)u_s f''(Y_s)\,ds\right] \\ &+ \frac{1}{2}\mathbb{E}\left[\int_0^t u_s\left(\frac{u_0 - u_s}{\beta}\right)^2 f^{(3)}(Y_s)ds\right] \\ &+ \frac{1}{2}\mathbb{E}\left[\int_0^t u_s\left(\frac{u_0 - u_s}{\beta}\right)^3 \int_0^1 (1-\theta)^2 f^{(4)}\left(Y_s + \theta\left(\frac{1}{\beta}(u_0 - u_s)\right)\right)d\theta ds\right] \\ &- \mu\mathbb{E}\left[\int_0^t f'(Y_s)\,ds\right] - \frac{\sigma^2}{2}\mathbb{E}\left[\int_0^t f''(Y_s)\,ds\right]. \end{aligned} \tag{2.4}$$

The local martingale term is actually a martingale by condition ($H_{Forward}$).

For a smooth enough function g and $p > 1$, we have the formula (A.13) shown in the Appendix 3

$$\begin{aligned} \int_0^t g(Y_s)u_s^p\,ds &= \frac{1 - e^{-p\beta t}}{p\beta}g(x_0)u_0^p - \frac{\mu}{p\beta}\int_0^t e^{-p\beta(t-r)}u_r^p g'(Y_r)\,dr + \frac{\mu}{p\beta}\int_0^t u_r^p g'(Y_r)\,dr \\ &+ \sigma\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^p g'(Y_r)\,dW_r ds - \frac{\sigma^2}{2p\beta}\int_0^t e^{-p\beta(t-r)}u_r^p g''(Y_r)\,dr + \frac{\sigma^2}{2p\beta}\int_0^t u_r^p g''(Y_r)\,dr \\ &- \mu\int_0^t e^{-p\beta(t-r)}u_r^{p-1}g(Y_r)\,dr + \mu\int_0^t u_r^{p-1}g(Y_r)\,dr - \frac{(p-1)\sigma^2\beta}{2}\int_0^t e^{-p\beta(t-r)}u_r^{p-2}g(Y_r)\,dr \\ &+ \frac{(p-1)\sigma^2\beta}{2}\int_0^t u_r^{p-2}g(Y_r)\,dr + \beta\sigma p\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-1}g(Y_r)\,dW_r ds \\ &- \sigma^2\int_0^t e^{-p\beta(t-r)}u_r^{p-1}g'(Y_r)\,dr + \sigma^2\int_0^t u_r^{p-1}g'(Y_r)\,dr\,. \end{aligned} \tag{2.5}$$

This formula is also valid for $p = 1$ by eliminating the terms that are multiplied by $p - 1$.

Lemma 2.3. *For $g \in \mathcal{C}_b^2$, we have the following controls:*

(i) $\left|\mathbb{E}\left[\int_0^t g(Y_s)u_s\, ds\right]\right| \le C_{\mu,g,g',g'',\sigma,t}$

(ii) $\left|\mathbb{E}\left[\int_0^t g(Y_s)u_s^2\, ds\right]\right| \le C_{\mu,g,\sigma,t}\beta\,.$

(iii) $\left|\mathbb{E}\left[\int_0^t g(Y_s)u_s^3\, ds\right]\right| \le C_{\mu,g,g',g'',\sigma,t}\beta\,.$

Proof.

(i). We take the expectation in formula (2.5), for $p = 1$

$$\begin{aligned}\mathbb{E}\left[\int_0^t g(Y_s)u_s\, ds\right] &= \frac{1-e^{-\beta t}}{\beta}g(x_0)u_0 - \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r g'(Y_r)\, dr\right] + \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t u_r g'(Y_r)\, dr\right]\\ &- \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r g''(Y_r)\, dr\right] + \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t u_r g''(Y_r)\, dr\right] - \mu\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}g(Y_r)\, dr\right]\\ &+ \mu\mathbb{E}\left[\int_0^t g(Y_r)\, dr\right] - \sigma^2\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}g'(Y_r)\, dr\right] + \sigma^2\mathbb{E}\left[\int_0^t g'(Y_r)\, dr\right].\end{aligned} \tag{2.6}$$

By using the control on the expectation of $\sup_{s\le r}|u_s|$ in (1.12) and the fact that $g \in \mathcal{C}_b^2$

$$\begin{aligned}&\left|\frac{\mu}{\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r g'(Y_r)\, dr\right]\right| \le \frac{\mu}{\beta}\|g'\|_{L^\infty}\int_0^t e^{-\beta(t-r)}\mathbb{E}|u_r|\, dr\\ &\le C_{\mu,g',t}\frac{1}{\sqrt{\beta}}\int_0^t e^{-\beta(t-r)}\, dr \le C_{\mu,g',t}\frac{1}{\beta\sqrt{\beta}}\end{aligned} \tag{2.7}$$

and similarly,

$$\left|\frac{\mu}{\beta}\mathbb{E}\left[\int_0^t u_r g'(Y_r)\, dr\right]\right| \le C_{\mu,g',t}\frac{1}{\sqrt{\beta}}\,; \quad \left|\frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r g''(Y_r)\, dr\right]\right| \le C_{\mu,g'',\sigma,t}\frac{1}{\beta\sqrt{\beta}}\,; \tag{2.8}$$

$$\left|\frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t u_r g''(Y_r)\, dr\right]\right| \le C_{\mu,g'',\sigma,t}\frac{1}{\sqrt{\beta}}\,; \quad \left|\mu\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}g(Y_r)\, dr\right]\right| \le C_{\mu,g,t}\frac{1}{\beta}\,; \tag{2.9}$$

$$\left|\mu\mathbb{E}\left[\int_0^t g(Y_r)\, dr\right]\right| \le C_{\mu,g,t}\,; \quad \left|\sigma^2\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}g'(Y_r)\, dr\right]\right| \le C_{\mu,g',\sigma,t}\frac{1}{\beta}\,; \tag{2.10}$$

and finally

$$\left|\sigma^2\mathbb{E}\left[\int_0^t g'(Y_r)\, dr\right]\right| \le C_{\mu,g',\sigma,t}\,. \tag{2.11}$$

We can conclude that we can bound uniformly in β:

$$\left|\mathbb{E}\left[\int_0^t g(Y_s)u_s\, ds\right]\right| \le C_{\mu,g,g',g'',\sigma,t}\,, \tag{2.12}$$

where $C_{\mu,g,g',g'',\sigma,t}$ does not depend on β.

***(ii)*.** For a second control, we take the control on the second moment of $(u_t)_{t\geq 0}$ in (A.5) to obtain that

$$\left|\mathbb{E}\left[\int_0^t g(Y_s)u_s^2\,ds\right]\right| \leq \|g\|_{L^\infty(\mathbb{R})}\int_0^t \mathbb{E}\left[u_s^2\right]\,ds \leq C_{\mu,g,\sigma,t}\beta\,. \tag{2.13}$$

***(iii)*.** We take the expectation in formula (2.5), for $p=3$

$$\begin{aligned}
\int_0^t g(Y_s)u_s^3\,ds &= \frac{1-e^{-3\beta t}}{3\beta}g(x_0)u_0^3 - \frac{\mu}{3\beta}\int_0^t e^{-3\beta(t-r)}u_r^3 g'(Y_r)\,dr + \frac{\mu}{3\beta}\int_0^t u_r^3 g'(Y_r)\,dr\\
&+\sigma\int_0^t\int_0^s e^{-3\beta(s-r)}u_r^3 g'(Y_r)\,dW_r ds - \frac{\sigma^2}{6\beta}\int_0^t e^{-3\beta(t-r)}u_r^3 g''(Y_r)\,dr + \frac{\sigma^2}{6\beta}\int_0^t u_r^3 g''(Y_r)\,dr\\
&-\mu\int_0^t e^{-3\beta(t-r)}u_r^2 g(Y_r)\,dr + \mu\int_0^t u_r^2 g(Y_r)\,dr - \sigma^2\beta\int_0^t e^{-3\beta(t-r)}u_r g(Y_r)\,dr\\
&+\sigma^2\beta\int_0^t u_r g(Y_r)\,dr + 3\beta\sigma\int_0^t\int_0^s e^{-3\beta(s-r)}u_r^2 g(Y_r)\,dW_r ds\\
&-\sigma^2\int_0^t e^{-3\beta(t-r)}u_r^2 g'(Y_r)\,dr + \sigma^2\int_0^t u_r^2 g'(Y_r)\,dr\,.
\end{aligned} \tag{2.14}$$

By using the various bounds on the first, second moment and absolute value of the third power of $(u_t)_{t\geq 0}$ in (A.5) and (A.6) and the previous bounds of this lemma in *(i)* and *(ii)*, we obtain the required bound. This concludes the proof of the lemma.

■

Recalling the equality (2.4)

$$\begin{aligned}
\mathbb{E}f(x_t) - \mathbb{E}f(Y_t) &= \mathbb{E}\left[\int_0^t f'(Y_s)u_s\,ds\right] + \frac{1}{\beta}\mathbb{E}\left[\int_0^t (u_0-u_s)\,u_s f''(Y_s)\,ds\right]\\
&+\frac{1}{2}\mathbb{E}\left[\int_0^t u_s\left(\frac{u_0-u_s}{\beta}\right)^2 f^{(3)}(Y_s)ds\right]\\
&+\frac{1}{2}\mathbb{E}\left[\int_0^t u_s\left(\frac{u_0-u_s}{\beta}\right)^3\int_0^1(1-\theta)^2 f^{(4)}\left(Y_s+\theta\left(\frac{1}{\beta}(u_0-u_s)\right)\right)\,d\theta ds\right]\\
&-\mu\mathbb{E}\left[\int_0^t f'(Y_s)\,ds\right] - \frac{\sigma^2}{2}\mathbb{E}\left[\int_0^t f''(Y_s)\,ds\right],
\end{aligned} \tag{2.15}$$

we analyse each of these terms separately by using the general formula (2.5) and the controls in Lemma 2.3.

We consider $g=f'$ and $p=1$ in equation (2.5) to obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f'(Y_s)u_s\,ds\right] &= \frac{1-e^{-\beta t}}{\beta}f'(x_0)u_0 - \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r f''(Y_r)\,dr\right]\\
&+\frac{\mu}{\beta}\mathbb{E}\left[\int_0^t u_r f''(Y_r)\,dr\right] - \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r f^{(3)}(Y_r)\,dr\right] + \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t u_r f^{(3)}(Y_r)\,dr\right]\\
&-\mu\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}f'(Y_r)\,dr\right] + \mu\mathbb{E}\left[\int_0^t f'(Y_r)\,drds\right]\\
&-\sigma^2\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}f''(Y_r)\,dr\right] + \sigma^2\mathbb{E}\left[\int_0^t f''(Y_r)\,dr\right].
\end{aligned} \tag{2.16}$$

The second term of the r.h.s. of the equation (2.15) is rewritten as

$$\frac{1}{\beta}\mathbb{E}\left[\int_0^t (u_0-u_s)\,u_s f''(Y_s)\,ds\right] = \frac{u_0}{\beta}\mathbb{E}\left[\int_0^t u_s f''(Y_s)\,ds\right] - \frac{1}{\beta}\mathbb{E}\left[\int_0^t u_s^2 f''(Y_s)\,ds\right] \tag{2.17}$$

and by taking $g = f''$ and $p = 2$, we obtain from (2.5) that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f''(Y_s)u_s^2\,ds\right] &= \frac{1-e^{-2\beta t}}{2\beta} f''(x_0)u_0^2 - \frac{\mu}{2\beta}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r^2 f^{(3)}(Y_r)\,dr\right] \\
&+ \frac{\mu}{2\beta}\mathbb{E}\left[\int_0^t u_r^2 f^{(3)}(Y_r)\,dr\right] - \frac{\sigma^2}{4\beta}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r^2 f^{(4)}(Y_r)\,dr\right] + \frac{\sigma^2}{4\beta}\mathbb{E}\left[\int_0^t u_r^2 f^{(4)}(Y_r)\,dr\right] \\
&- \mu\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r f''(Y_r)\,dr\right] + \mu\mathbb{E}\left[\int_0^t u_r f''(Y_r)\,dr\right] - \frac{\sigma^2\beta}{2}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)} f''(Y_r)\,dr\right] \\
&+ \frac{\sigma^2\beta}{2}\mathbb{E}\left[\int_0^t f''(Y_r)\,dr\right] - \sigma^2\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r f^{(3)}(Y_r)\,dr\right] + \sigma^2\mathbb{E}\left[\int_0^t u_r f^{(3)}(Y_r)\,dr\right].
\end{aligned} \tag{2.18}$$

By combining these terms and simplifying, we obtain that (2.15) can be rewritten as

$$\begin{aligned}
\mathbb{E}f(x_t) - \mathbb{E}f(Y_t) &= \frac{1-e^{-\beta t}}{\beta} f'(x_0)u_0 - \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r f''(Y_r)\,dr\right] + \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t u_s f''(Y_s)\,ds\right] \\
&- \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t e^{-\beta(t-r)}u_r f^{(3)}(Y_r)\,dr\right] + \frac{\sigma^2}{2\beta}\mathbb{E}\left[\int_0^t u_r f^{(3)}(Y_r)\,dr\right] - \mu\mathbb{E}\left[\int_0^t e^{-\beta(t-r)} f'(Y_r)\,dr\right] \\
&- \sigma^2\mathbb{E}\left[\int_0^t e^{-\beta(t-r)} f''(Y_r)\,dr\right] + \frac{u_0}{\beta}\mathbb{E}\left[\int_0^t u_r f''(Y_r)\,dr\right] - \frac{1-e^{-2\beta t}}{2\beta^2} f''(x_0)u_0^2 \\
&+ \frac{\mu}{2\beta^2}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r^2 f^{(3)}(Y_r)\,dr\right] - \frac{\mu}{2\beta^2}\mathbb{E}\left[\int_0^t u_r^2 f^{(3)}(Y_r)\,dr\right] + \frac{\sigma^2}{4\beta^2}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r^2 f^{(4)}(Y_r)\,dr\right] \\
&- \frac{\sigma^2}{4\beta^2}\mathbb{E}\left[\int_0^t u_r^2 f^{(4)}(Y_r)\,dr\right] + \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r f''(Y_r)\,dr\right] - \frac{\mu}{\beta}\mathbb{E}\left[\int_0^t u_r f''(Y_r)\,dr\right] \\
&+ \frac{\sigma^2}{2}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)} f''(Y_r)\,dr\right] + \frac{\sigma^2}{\beta}\mathbb{E}\left[\int_0^t e^{-2\beta(t-r)}u_r f^{(3)}(Y_r)\,dr\right] - \frac{\sigma^2}{\beta}\mathbb{E}\left[\int_0^t u_r f^{(3)}(Y_r)\,dr\right] \\
&+ \frac{1}{2\beta^2}\mathbb{E}\left[\int_0^t \left(u_s^3 + u_0^2 u_s - 2u_0 u_s^2\right) f^{(3)}(Y_s)ds\right] \\
&- \frac{1}{2\beta^3}\mathbb{E}\left[\int_0^t (u_s^4 - 3u_0u_s^3 + 3u_0^2u_s^2 - u_0^3u_s)\int_0^1 (1-\theta)^2 f^{(4)}\left(Y_s + \theta\left(\frac{1}{\beta}(u_0-u_s)\right)\right)\,d\theta ds\right].
\end{aligned} \tag{2.19}$$

We set aside for the moment the last two terms of the r.h.s. that were obtained by the Taylor expansion (2.3). For the rest of the terms in the r.h.s, it can be easily seen that, by applying the generic bounds (2.12) and (2.13) from Lemma 2.3, with different values for the function g that will depend on the various derivatives of f, they are bounded by $\frac{C_{\mu,\sigma,f,\ldots,f^{(6)},x_0,u_0,t}}{\beta}$, where $C_{\mu,\sigma,f,\ldots,f^{(6)},x_0,u_0,t}$ depends solely μ, σ, the bounds of the test function and its first 6 derivatives, the initial values x_0 and u_0 and time t.

Turning towards the Taylor expansion terms, for the penultimate expectation we consider the bounds of the Lemma 2.3, which allow us to obtain a bound in $\frac{1}{\beta}$ provided f is 5 times differentiable with bounded derivatives. Concerning the last term, we recall that the fourth derivative of f is bounded. By applying the controls of order $1/2$ in β for the absolute value, of order 1 for the second moment, of order $3/2$ for the absolute value of the third power and of order 2 for the fourth moment of $(u_t)_{t\geq 0}$, all shown in Appendix A, we obtain again a bound in $\frac{1}{\beta}$ for the last term of the expansion (2.19) and we conclude on the lemma.

■

Thus we showed in a toy example, where the drift μ is a constant, that we can control the weak error in $\frac{1}{\beta}$, provided the test function has sufficient regularity. This result was proven using Taylor expansions, so controlling the moments of the velocity component of the Langevin process, like in Lemma 2.3, is essential. An important argument in obtaining the the linear decrease in β of the error, is that the even moments, say $2k$ for any non-negative integer k, are controlled as β^k while the odd moments, say $2k+1$, are also controlled as β^k.

2.2 General drift

The same techniques used in the previous subsection Toy example: Constant drift case do not generalise well in the case of a non-constant drift, so a technique based on the mild-equations verified by the density of the processes is considered. Once more we utilise Taylor expansions so controlling the various moments in the velocity component of the Langevin process, with the appropriate power of β is essential.

Mild Equation

We introduce the mild equations for the densities of the Langevin process (1.6) and the elliptic diffusion (1.7).

Let $(\widetilde{x}_t, \widetilde{u}_t)_{t\geq 0}$ and $(\widetilde{Y}_t)_{t\geq 0}$ be the solutions to the equations (1.6) and (1.7) in the case where $\mu \equiv 0$. We define $\Gamma_{\mathrm{OU}} \colon \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ as the transition density of $(\widetilde{x}_t, \widetilde{u}_t)_{t\geq 0}$ meaning that $\Gamma_{\mathrm{OU}}(t; y, v; x, u) = \mathbb{P}_{y,v}\left((\widetilde{x}_t, \widetilde{u}_t) \in (dx, du)\right)/dxdu$ and $\Gamma_{\mathrm{B}} \colon \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ as the transition density of $(\widetilde{Y}_t)_{t\geq 0}$ such that $\Gamma_{\mathrm{B}}(t; y; x) = \dfrac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}(x-y)^2\right)$.

The semi-groups associated with the transition functions Γ_{OU} et Γ_{B} are denoted by $(S_t)_{t>0}$ and respectively $(\bar{S}_t)_{t>0}$, for any $f \in \mathcal{C}_b(\mathbb{R} \times \mathbb{R})$ and $g \in \mathcal{C}_b(\mathbb{R})$ where

$$S_t(f)(y, v) = \mathbb{E}_{y,v}\left[f(\widetilde{x}_t, \widetilde{u}_t)\right] = \int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t; y, v; x, u) f(x, u)\, dxdu$$

and

$$\bar{S}_t(g)(y) = \mathbb{E}_y\left[g(\widetilde{Y}_t)\right] = \int_{\mathbb{R}} \Gamma_{\mathrm{B}}(t; y; x) g(x)\, dx\,.$$

We also introduce the following functionals

$$S_t^*(\mu)(x, u) = \int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t; y, v; x, u)\mu(dy, dv) \tag{2.20}$$

and

$$S_t'(f)(x, u) = \int_{\mathbb{R}\times\mathbb{R}} \partial_v \Gamma_{\mathrm{OU}}(t; y, v; x, u) f(y, v)\, dydv\,. \tag{2.21}$$

Similarly we denote by $\bar{S}^*$ and $\bar{S}'$ the equivalent functionals associated to the diffusion process $(\widetilde{Y}_t)_{t\geq 0}$.

We have that for all $t > 0$ and $f \in \mathcal{C}_c(\mathbb{R} \times \mathbb{R})$, the function $H_{t,f}$ defined as

$$H_{t,f} \colon (s, y, v) \in [0, t) \times \mathbb{R} \times \mathbb{R} \mapsto S_{t-s}(f)(y, v)$$

is the classical solution of the PDE

$$\begin{cases} \partial_s H_{t,f} + \left(v\partial_y - \beta v \partial_v + \dfrac{\beta^2\sigma^2}{2}\partial_{vv} \right) H_{t,f} = 0 & \text{on } [0,t) \times \mathbb{R} \times \mathbb{R} \\ \lim\limits_{s\to t^-} H_{t,f}(s,y,v) = f(y,v), & \text{on } \mathbb{R} \times \mathbb{R}\,. \end{cases} \tag{2.22}$$

By Ito's formula, we have

$$\begin{aligned} \mathbb{E}_{\mu_0}\left[H_{t,f}(t,x_t,u_t)\right] &= \mathbb{E}_{\mu_0}\left[H_{t,f}(0,x_0,u_0)\right] + \int_0^t \mathbb{E}_{\mu_0}\left[\left(\partial_s + \left(u_s\partial_y - \beta u_s \partial_v + \frac{\beta^2\sigma^2}{2}\partial_{vv}\right)\right) H_{t,f}(s,x_s,u_s)\right] ds \\ &\quad + \int_0^t \mathbb{E}_{\mu_0}\left[\beta\mu(x_s)\partial_v H_{t,f}(s,x_s,u_s)\right] ds \\ &= \int_{\mathbb{R}\times\mathbb{R}} H_{t,f}(0,x_0,u_0)\,\mu_0(dx_0,du_0) + \beta \int_0^t \int_{\mathbb{R}^2} \mu(y)\partial_v H_{t,f}(s,y,v)\rho(s,y,v)\,dsdydv\,. \end{aligned} \tag{2.23}$$

The first term of the r.h.s. is rewritten using Fubini's theorem, as

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}} H_{t,f}(0,x_0,u_0)\,\mu_0(dx_0,du_0) = \int_{\mathbb{R}\times\mathbb{R}} S_t(f)(x_0,u_0)\,\mu_0(dx_0,du_0) \\ &= \int_{\mathbb{R}^4} \Gamma_{\mathrm{OU}}(t;x_0,u_0;x,u) f(x,u)\,dxdu\mu_0(dx_0,du_0) = \int_{\mathbb{R}^2} S_t^*(\mu_0)(x,u) f(x,u)\,dxdu\,. \end{aligned} \tag{2.24}$$

Let $(\rho_t)_{t\geq 0}$ be the time-marginal densities of a solution to the SDE (1.6). For the second term, we have that

$$\begin{aligned} &\beta \int_0^t \int_{\mathbb{R}^2} \mu(y)\partial_v H_{t,f}(s,y,v)\rho(s,y,v)\,dsdydv = \beta \int_0^t \int_{\mathbb{R}^2} \mu(y)\rho(s,y,v)\partial_v S_{t-s}(f)(y,v)\,dsdydv \\ &= \int_0^t \int_{\mathbb{R}^2} f(x,u)\,dsdxdu \int_{\mathbb{R}^2} \beta\mu(y)\rho(s,y,v)\partial_v \Gamma_{\mathrm{OU}}(t;y,v;x,u)\,dydv \\ &= \int_0^t \int_{\mathbb{R}^2} f(x,u) S'_{t-s}\left(\beta\rho(s,\cdot,\cdot)\mu(\cdot)\right)\,dsdxdu\,. \end{aligned} \tag{2.25}$$

According to PDE (2.22), we have that $H_{t,f}(t,\cdot,\cdot) = f(\cdot,\cdot)$. Then the equation (2.23) becomes

$$\int_{\mathbb{R}^2} f(x,u)\rho(t,x,u)\,dxdu = \int_{\mathbb{R}^2} S_t^*(\mu_0)(x,u) f(x,u)\,dxdu + \int_0^t \int_{\mathbb{R}^2} f(x,u) S'_{t-s}\left(\beta\rho(s,\cdot,\cdot)\mu(\cdot)\right)\,dsdxdu\,. \tag{2.26}$$

Thus for any $f \in \mathcal{C}_c(\mathbb{R}\times\mathbb{R})$:

$$\int_{\mathbb{R}^2} f(x,u)\left(\rho(t,x,u) - S_t^*(\mu_0)(x,u) - \int_0^t S'_{t-s}\left(\beta\rho(s,\cdot,\cdot)\mu(\cdot)\right)\,ds\right)\,dxdu = 0\,. \tag{2.27}$$

We conclude that the time marginal of the process $(x_t,u_t)_{t\geq 0}$, that verifies the SDE (1.6) with initial condition μ_0, verifies the mild equation:

$$\rho(t,x,u) = S_t^*(\mu_0)(x,u) + \int_0^t S'_{t-s}\left(\rho(s,\cdot,\cdot)\beta\mu(\cdot)\right)\,ds \tag{2.28}$$

Remark 2.4. *The time-marginal densities of a solution to the SDE* (1.6) *has bounded tails (with bounds that depend on β) since for fixed initial conditions, the solution is a Gaussian process, and μ_0 vanishes at infinity. In the Appendix, in Lemma 4.5, we show that ρ and $\partial_x\rho$ are bounded. These bounds are useful when applying various integration by parts against functions that vanish at infinity.*

By following a similar procedure we can conclude that the time marginal of the process $(Y_t)_{t\geq 0}$, that verifies the SDE (1.7) with initial condition $\mu_0^Y = \int_{\mathbb{R}} \mu_0(\cdot, dv)$, verifies

$$p(t,x) = \bar{S}_t^*(\mu_0^Y)(x) + \int_0^t \bar{S}'_{t-s}\left(p(s,\cdot)\mu(\cdot)\right)\, ds\,. \tag{2.29}$$

Theorem 2.5. *Assume ($H_{Forward}$) is verified and assume ρ and p are solutions to the mild equations* (2.28) *and, respectively,* (2.29)*. Then, for large enough β, we have that*

$$\left\| \int_{\mathbb{R}} \rho(t,x,u)\, du - p(t,x) \right\|_{L^1(\mathbb{R})} \leq C_{\mu_0,\mu,\sigma,t} \frac{\ln(\beta)}{\beta}\,. \tag{2.30}$$

Proof. We obtain the term to be bounded in (2.30)

$$\begin{aligned}\int_{\mathbb{R}} \rho(t,x,u)\, du &= \int_{\mathbb{R}\times\mathbb{R}} \left(\int_{\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\, du \right) \mu_0(dy,dv) \\ &+ \beta \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \partial_v \left(\int_{\mathbb{R}} \Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\, du \right) \rho(s,y,v)\mu(y)\, dydvds\,.\end{aligned} \tag{2.31}$$

We denote by M the marginal density of the position process

$$M(t;y,v;x) := \left(\int_{\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\, du \right) = \frac{1}{\sqrt{2\pi}\Sigma_{xx}(t)} \exp\left(-\frac{1}{2\Sigma_{xx}^2(t)} \left(x - y - \frac{v}{\beta}(1-e^{-\beta t}) \right)^2 \right) \tag{2.32}$$

so:

$$\partial_v M(t;y,v;x) = \frac{1-e^{-\beta t}}{\beta \Sigma_{xx}^2(t)} \left(x - y - \frac{v}{\beta}(1-e^{-\beta t}) \right) M(t;y,v;x)\,,$$

with Σ_{xx} defined in (A.4) as

$$\Sigma_{xx}^2(t) = \sigma^2 \left(t - \frac{2}{\beta}(1-e^{-\beta t}) + \frac{1}{2\beta}(1-e^{-2\beta t}) \right)\,.$$

Thus:

$$\begin{aligned}\int_{\mathbb{R}} \rho(t,x,u)\, du - p(t,x) &= \int_{\mathbb{R}} S_t^*(\mu_0)(x,u)\, du - \bar{S}_t^*(\mu_0^Y)(x) \\ &+ \beta \int_0^t \int_{\mathbb{R}^2} \frac{1-e^{-\beta(t-s)}}{\beta\Sigma_{xx}^2(t-s)} \left(x - y - \frac{v}{\beta}(1-e^{-\beta(t-s)}) \right) M(t-s;y,v;x)\rho(s,y,v)\mu(y)\, dydvds \\ &- \int_0^t \int_{\mathbb{R}} \partial_y \Gamma_{\mathrm{B}}(t-s;y;x) p(s,y)\mu(y)\, dyds\,.\end{aligned} \tag{2.33}$$

We rewrite this equation as

$$
\begin{aligned}
&\int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x) = \int_{\mathbb{R}} S_t^*(\mu_0)(x,u)\,du - \bar{S}_t^*(\mu_0)(x) \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \mu(y) \left(\frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}^2(t-s)} \left(x-y-\frac{v}{\beta}(1-e^{-\beta(t-s)}) \right) M(t-s;y,v;x) - \partial_y \Gamma_{\mathrm{B}}(t-s;y;x) \right) \rho(s,y,v)\,dydvds \\
&\quad + \int_0^t \int_{\mathbb{R}} \partial_y \Gamma_{\mathrm{B}}(t-s;y;x)\mu(y) \left(\int_{\mathbb{R}} \rho(s,y,v)\,dv - p(s,y) \right) dyds\,.
\end{aligned}
\tag{2.34}
$$

In order to obtain a bound as presented in (2.30), we utilise a Gronwall inequality on (2.34). On the right-hand side, we have the sum of three differences. The first difference corresponds to the initial terms of the mild equation. The second term represents the difference between the two different kernels of the mild equation, and finally, the third term allows to perform Gronwall's inequality. We analyse each of these terms separately.

Bounding the initial terms

Denote by $g(\mu;\sigma^2;x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$ the Gaussian probability density with mean μ and standard deviation σ.

We consider the first difference in the r.h.s. of equation (2.34) which corresponds to the initial value

$$
\begin{aligned}
&\int_{\mathbb{R}} S_t^*(\mu_0)(x,u)\,du - \bar{S}_t^*(\mu_0^Y)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\mu_0(dy,dv)du - \int_{\mathbb{R}} \Gamma_{\mathrm{B}}(t;y;x) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&= \int_{\mathbb{R}\times\mathbb{R}} g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \Sigma_{xx}^2(t), x \right) \mu_0(dy,dv) - \int_{\mathbb{R}} g(y,\sigma^2 t, x) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&= \int_{\mathbb{R}\times\mathbb{R}} g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \Sigma_{xx}^2(t), x \right) \mu_0(dy,dv) - \int_{\mathbb{R}} g(y,\sigma^2 t, x) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&= \int_{\mathbb{R}\times\mathbb{R}} g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \sigma^2 t, x \right) \mu_0(dy,dv) - \int_{\mathbb{R}} g(y,\sigma^2 t, x) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}} g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \Sigma_{xx}^2(t), x \right) \mu_0(dy,dv) \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}} g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \sigma^2 t, x \right) \mu_0(dy,dv)\,.
\end{aligned}
\tag{2.35}
$$

It is easy to see that $\Sigma_{xx}^2(t) < \sigma^2 t$, for any $t > 0, \beta > 0$.

Since the Gaussian densities are not degenerate, we have that g is a smooth function, so

$$
g\left(y + \frac{v}{\beta}(1-e^{-\beta t}), \sigma^2 t, x \right) = g(y,\sigma^2 t, x) + \frac{v}{\beta}(1-e^{-\beta t}) \int_0^1 \partial_\mu g\left(y + \theta\frac{v}{\beta}(1-e^{-\beta t}), \sigma^2 t, x \right) d\theta
\tag{2.36}
$$

then by taking the difference, integrating and taking the norm we have that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}\times\mathbb{R}} g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t, x\right)\mu_0(dy,dv) - \int_{\mathbb{R}} g(y,\sigma^2 t,x)\int_{\mathbb{R}}\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})}\\
&\leq \left\| \int_{\mathbb{R}^2} \frac{v}{\beta}(1-e^{-\beta t})\int_0^1 \partial_\mu g\left(y+\theta\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t,x\right)\,\mu_0(dy,dv)d\theta\right\|_{L^1(\mathbb{R})}\\
&\leq \left\| \int_{\mathbb{R}}\int_0^1 \frac{v}{\beta}(1-e^{-\beta t})\left(\int_{\mathbb{R}} \partial_\mu g\left(y+\theta\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t,x\right)\mu_0(y,v)\,dy\right)dvd\theta\right\|_{L^1(\mathbb{R})} \leq \frac{1}{\beta}\left\|v\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)}
\end{aligned} \tag{2.37}$$

where for the last inequality and integration by parts has been performed with boundary terms that equal 0 since $g(y,\cdot,\cdot)\to 0$ as $|y|\to+\infty$. We also have that

$$\begin{aligned}
&g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\Sigma_{xx}^2(t),x\right) = g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t,x\right)\\
&\quad+(\Sigma_{xx}(t)-\sigma\sqrt{t})\int_0^1 \partial_\sigma g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),(\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t})))^2,x\right)d\theta
\end{aligned}$$

thus by the following property of Gaussian densities $\partial_\sigma g = \sigma\partial^2_{\mu\mu} g$

$$\begin{aligned}
&\left\|\int_{\mathbb{R}\times\mathbb{R}}\left(g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\Sigma_{xx}^2(t),x\right)-g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t,x\right)\right)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq \left|\Sigma_{xx}(t)-\sigma\sqrt{t}\right|\left\|\int_{\mathbb{R}\times\mathbb{R}}\int_0^1 \partial_\sigma g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),(\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t})))^2,x\right)\mu_0(dy,dv)d\theta\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq \left|\Sigma_{xx}(t)-\sigma\sqrt{t}\right|\left\|\int_{\mathbb{R}\times\mathbb{R}}\int_0^1 \left[Z\partial^2_{yy} g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),Z^2,x\right)\right]_{Z=\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t}))}\mu_0(dy,dv)d\theta\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq \left|\Sigma_{xx}(t)-\sigma\sqrt{t}\right|\left\|\int_{\mathbb{R}\times\mathbb{R}}\int_0^1 \left[Z g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),Z^2,x\right)\right]_{Z=\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t}))}\partial^2_{yy}\mu_0(dy,dv)d\theta\right\|_{L^1(\mathbb{R}\times\mathbb{R})}.
\end{aligned} \tag{2.38}$$

The boundary terms obtained from the various i.b.p. are zero since $g(y,\cdot,\cdot)$ and $\partial_y g(y,\cdot,\cdot)$ go to zero as $|y|\to+\infty$.

Since $\Sigma^2_{xx}(t)<\sigma^2 t$, for any $t>0,\beta>0$, then we have that for any $\theta\in[0,1]$, $\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t}))\leq\sigma\sqrt{t}$ thus

$$\begin{aligned}
&\left\|\int_{\mathbb{R}\times\mathbb{R}}\left(g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\Sigma_{xx}^2(t),x\right)-g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),\sigma^2 t,x\right)\right)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq \sigma\sqrt{t}\left|\Sigma_{xx}(t)-\sigma\sqrt{t}\right|\int_{\mathbb{R}\times\mathbb{R}}\left|\partial^2_{yy}\mu_0(dy,dv)\right|\int_0^1 d\theta\int_{\mathbb{R}} g\left(y+\frac{v}{\beta}(1-e^{-\beta t}),(\sigma\sqrt{t}+\theta((\Sigma_{xx}(t)-\sigma\sqrt{t})))^2,x\right)dx\\
&\leq \sigma\sqrt{t}\left|\Sigma_{xx}(t)-\sigma\sqrt{t}\right|\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\leq\sigma\sqrt{t}\frac{\sigma^2 t-\Sigma^2_{xx}(t)}{\Sigma_{xx}(t)+\sigma\sqrt{t}}\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq\frac{\frac{2}{\beta}(1-e^{-\beta t})-\frac{1}{2\beta}(1-e^{-2\beta t})}{\sigma\sqrt{t}}\sigma\sqrt{t}\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\\
&\leq\frac{2\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}}{\beta}.
\end{aligned}$$

we can conclude that

$$\left\|\int_{\mathbb{R}} S_t^*(\mu_0)(x,u)\,du-\bar{S}_t^*(\mu_0^Y)(x)\right\|_{L^1(\mathbb{R})}\leq\frac{1}{\beta}\left(\left\|v\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)}+2\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}\right). \tag{2.39}$$

Bounding time integral

We consider the middle term of (2.34)

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \mu(y) \left(\frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}^2(t-s)} \left(x - y - \frac{v}{\beta}(1-e^{-\beta(t-s)}) \right) M(t-s; y, v; x) - \partial_y \Gamma_{\mathrm{B}}(t-s; y; x) \right) \rho(s,y,v)\, dydvds \\
&= \int_0^t \int_{\mathbb{R}^2} \mu(y) \left((1-e^{-\beta(t-s)}) \partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \Sigma_{xx}^2(t-s), x) - \partial_y g(y, \sigma^2(t-s), x) \right) \rho(s,y,v)\, dydvds \\
&= \int_0^t \int_{\mathbb{R}^2} \mu(y) \left((1-e^{-\beta(t-s)}) \partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x) - \partial_y g(y, \sigma^2(t-s), x) \right) \rho(s,y,v)\, dydvds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \mu(y)(1-e^{-\beta(t-s)}) \rho(s,y,v)\, dydvds \times \\
&\qquad \times \left(\partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \Sigma_{xx}^2(t-s), x) - \partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x) \right).
\end{aligned}
\tag{2.40}
$$

This term can therefore be written as the sum of two differences. The first difference is between two Gaussian densities with different means but same variance and the second difference is between Gaussian densities of different variances but same mean.

Difference between two ex-centred Gaussians in (2.40) Since the Gaussian density is a smooth function we can apply Taylor's formula up to order two, with integral remainder

$$
\begin{aligned}
&\partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x) = \partial_y g(y, \sigma^2(t-s), x) \\
&\quad + \frac{v}{\beta}(1-e^{-\beta(t-s)}) \partial_{yy} g\left(y, \sigma^2(t-s), x\right) \\
&\quad + \frac{v^2}{\beta^2}(1-e^{-\beta(t-s)})^2 \int_0^1 (1-\theta) \partial_{yyy} g\left(y + \theta \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x \right) d\theta
\end{aligned}
\tag{2.41}
$$

thus inserting this expansion into the first difference of the r.h.s. of equality (2.40) gives us

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \mu(y) \left((1-e^{-\beta(t-s)}) \partial_y g\left(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x \right) - \partial_y g(y, \sigma^2 t, x) \right) \rho(s,y,v)\, dydvds \\
&= -\int_0^t e^{-\beta(t-s)} \int_{\mathbb{R}} \mu(y) \partial_y g(y, \sigma^2(t-s), x) \left(\int_{\mathbb{R}} \rho(s,y,v)\, dv \right) dyds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \frac{v}{\beta} \mu(y)(1-e^{-\beta(t-s)})^2 \partial_{yy} g\left(y, \sigma^2(t-s), x\right) \rho(s,y,v)\, dvdyds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \frac{v^2}{\beta^2} \mu(y)(1-e^{-\beta(t-s)})^3 \int_0^1 (1-\theta) \partial_{yyy} g\left(y + \theta \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x \right) d\theta \rho(s,y,v)\, dydvds\,.
\end{aligned}
\tag{2.42}
$$

Once more, we break up this equality and analyse each of the three terms of the right-hand side separately.

First term of (2.42) We apply an integration by parts of the first terms and take the $L^1(\mathbb{R})$ norm to obtain

$$
\begin{aligned}
&\left\| \int_0^t e^{-\beta(t-s)} \int_{\mathbb{R}} g(y,\sigma^2(t-s),x) \left(\int_{\mathbb{R}} \partial_y \left(\mu(y)\rho(s,y,v)\right) dv \right) dyds \right\|_{L^1(\mathbb{R})} \\
&\leq \|\mu\|_{L^\infty(\mathbb{R})} \int_0^t e^{-\beta(t-s)} \|\partial_y \rho(s,\cdot,\cdot)\|_{L^1(\mathbb{R}^2)} \, ds + \frac{\|\partial_y \mu\|_{L^\infty}}{\beta} \\
&\leq \|\mu\|_{L^\infty(\mathbb{R})} C_{\mu_0,\mu,\sigma,t} \int_0^t e^{-\beta(t-s)} \, ds + \frac{\|\partial_y \mu\|_{L^\infty}}{\beta} \\
&\leq C_{\mu_0,\mu,\sigma,t} \frac{1}{\beta} \, ,
\end{aligned}
\tag{2.43}
$$

where $C_{\mu_0,\mu,\sigma,t}$ does not depend on β. The boundary terms from the i.b.p. are zero since $g(y,\cdot,\cdot) \to 0$ as $|y| \to \infty$ and ρ is bounded in y for fixed β as in the Remark 2.4. We have applied the bound on the norm of the partial derivative of the density obtained in Lemma 2.7.

Second Term of (2.42) We have that

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^2} \frac{v}{\beta} \mu(y)(1-e^{-\beta(t-s)})^2 \partial_{yy} g\left(y,\sigma^2(t-s),x\right) \rho(s,y,v) \, dvdyds \right| \\
&\leq \|\mu\|_{L^\infty} \frac{1}{\beta} \int_0^t (1-e^{-\beta(t-s)})^2 \int_{\mathbb{R}} \left|\partial_{yy} g\left(y,\sigma^2(t-s),x\right)\right| \left| \int_{\mathbb{R}} v\rho(s,y,v) \, dv \right| \, dyds
\end{aligned}
\tag{2.44}
$$

and by taking the L^1 norm and considering the bound on the first moment in Lemma 2.7-*(iii)* and the control of the resulting time integral in (A.23)

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^2} \frac{v}{\beta} \mu(y)(1-e^{-\beta(t-s)})^2 \partial_{yy} g\left(y,\sigma^2(t-s),x\right) \rho(s,y,v) \, dvdyds \right\|_{L^1(\mathbb{R})} \\
&\leq \frac{1}{\beta} \|\mu\|_{L^\infty} C_{\mu_0,\mu,\sigma,t} \int_0^t \frac{(1-e^{-\beta s})^2}{\sigma^2 s} \, ds \leq C_{\mu_0,\mu,\sigma,t} \frac{\ln(\beta)}{\beta} \, .
\end{aligned}
\tag{2.45}
$$

The bound $C_{\mu_0,\mu,\sigma,t}$ is obtained from the Lemma 2.7 and the calculation in (A.23) and does not depend on β.

Third Term of (2.42) We take the $L^1(\mathbb{R})-$norm

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^2} \frac{v^2}{\beta^2} \mu(y)(1-e^{-\beta(t-s)})^3 \int_0^1 (1-\theta)\partial_{yyy} g\left(y+\theta\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x\right) d\theta \rho(s,y,v) \, dydvds \right\|_{L^1} \\
&= \left\| \int_0^t \int_{\mathbb{R}^2} \frac{v^2}{\beta^2} (1-e^{-\beta(t-s)})^3 \int_0^1 (1-\theta)\partial_{yy} g\left(y+\theta\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x\right) d\theta \partial_y \left(\rho\mu(y)\right) \, dydvds \right\|_{L^1} \\
&\leq \frac{1}{\beta^2} \int_0^t (1-e^{-\beta(t-s)})^3 \left\|\partial_{yy} g(\cdot,\sigma^2(t-s),\cdot)\right\|_{L^1(\mathbb{R})} \left\| \int_{\mathbb{R}} v^2 \left|\partial_y(\rho(s,y,v)\mu(y))\right| \right\|_{L^1(\mathbb{R})} ds \\
&\leq \frac{2}{\beta^2} \int_0^t (1-e^{-\beta(t-s)})^3 \left\|\partial_{yy} g(\cdot,\sigma^2(t-s),\cdot)\right\|_{L^1(\mathbb{R})} \left\| \int_{\mathbb{R}} v^2 \left|\partial_y(\rho(s,y,v)\mu(y))\right| \right\|_{L^1(\mathbb{R})} ds \, .
\end{aligned}
\tag{2.46}
$$

The bounded tails of ρ and the fact that $\partial_{yy} g(y,\cdot,\cdot) \to 0$ at infinity, gives that the boundaries terms of the i.b.p. are zero.

By the bounds on the second moment and derivative of the second moment in Lemma 2.8, we have that:

$$\left\|\int_0^t\int_{\mathbb{R}^2}\frac{v^2}{\beta^2}\mu(y)(1-e^{-\beta(t-s)})^3\int_0^1(1-\theta)\partial_{yyy}g\left(y+\theta\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x\right)\,d\theta\rho(s,y,v)\,dydvds\right\|_{L^1}$$
$$\le C_{\mu_0,\mu,\sigma,t}\frac{1}{\beta}\int_0^t(1-e^{-\beta s})^3\frac{1}{\sigma^2 s}\,ds\le C_{\mu_0,\mu,\sigma,t}\frac{\ln(\beta)}{\beta}\,. \tag{2.47}$$

The bound $C_{\mu_0,\mu,\sigma,t}$ is obtained from the Lemma 2.8 and the calculation in (A.23).

This concludes that the difference between the two ex-centric Gaussians is controlled as:

$$\left\|\int_0^t\int_{\mathbb{R}^2}\mu(\cdot)\left((1-e^{-\beta(t-s)})\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x)-\partial_y g(y,\sigma^2(t-s),x)\right)\rho(\cdot,\cdot,\cdot)\,dydvds\right\|_{L^1(\mathbb{R})}$$
$$\le C_{\mu,\sigma,\mu_0,t}\frac{\ln(\beta)}{\beta}\,. \tag{2.48}$$

Difference between two Gaussians with different variances in (2.40) Let G be the cumulative distribution function of the centred Gaussian distribution with unit variance such that:

$$\partial_x G\left(\frac{x-\mu}{\sigma}\right)=-\partial_\mu G\left(\frac{x-\mu}{\sigma}\right)=g(\mu,\sigma^2,x)\,.$$

We have the variance terms in (2.40):

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\mu(y)(1-e^{-\beta(t-s)})\rho(s,y,v)\,dydvds\Big(\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\Sigma_{xx}^2(t-s),x)\\&\quad-\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x)\Big)\\&=\int_0^t\int_{\mathbb{R}^2}(1-e^{-\beta(t-s)})\partial_y\left(\mu(y)\rho(s,y,v)\right)\,dydvds\left(\partial_y G\left(\frac{z}{\Sigma_{xx}(t-s)}\right)-\partial_y G\left(\frac{z}{\sigma\sqrt{t-s}}\right)\right)\Big|_{z=x-y-\frac{v}{\beta}(1-e^{-\beta(t-s)})}\\&=-\int_0^t\int_{\mathbb{R}^2}(1-e^{-\beta(t-s)})\partial_{yy}\left(\mu(y)\rho(s,y,v)\right)\,dydvds\left(G\left(\frac{z}{\Sigma_{xx}(t-s)}\right)-G\left(\frac{z}{\sigma\sqrt{t-s}}\right)\right)\Big|_{z=x-y-\frac{v}{\beta}(1-e^{-\beta(t-s)})}\end{aligned} \tag{2.49}$$

then by taking the L^1-norm in x:

$$\begin{aligned}&\left\|\int_0^t\int_{\mathbb{R}^2}\mu(y)(1-e^{-\beta(t-s)})\rho(\cdot,\cdot,\cdot)\,dydvds\left(\partial_y g(0,\Sigma_{xx}^2(t-s),z)-\partial_y g(0,\sigma^2(t-s),z)\right)\Big|_{z=x-y-\frac{v}{\beta}(1-e^{-\beta(t-s)})}\right\|_{L^1(\mathbb{R})}\\&=\left\|\int_0^t\int_{\mathbb{R}^2}(1-e^{-\beta(t-s)})\partial_{yy}\left(\mu(y)\rho(\cdot,\cdot,\cdot)\right)\,dydvds\left(G\left(\frac{z}{\Sigma_{xx}(t-s)}\right)-G\left(\frac{z}{\sigma\sqrt{t-s}}\right)\right)\Big|_{z=x-y-\frac{v}{\beta}(1-e^{-\beta(t-s)})}\right\|_{L^1(\mathbb{R})}\\&\le\int_0^t(1-e^{-\beta(t-s)})\left\|\partial_{yy}\left(\mu(y)\rho(\cdot,\cdot,\cdot)\right)\right\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}}\left|\left(G\left(\frac{z}{\Sigma_{xx}(t-s)}\right)-G\left(\frac{z}{\sigma\sqrt{t-s}}\right)\right)\right|\,dzds\,.\end{aligned} \tag{2.50}$$

The various boundary terms obtained from the i.b.p. are null since the difference of Gaussian densities and cdfs with different variances go to 0 as $|y|\to\infty$ while ρ and $\partial_y\rho$ are bounded as in the Remark 2.4.

Let $\sigma_1, \sigma 2$ be two strictly positive reals, since the cdf of a non-degenerate Gaussian random variable is a smooth function, we have that:

$$
\begin{aligned}
&G\left(\frac{z}{\sigma_2}\right) - G\left(\frac{z}{\sigma_1}\right) \\
&= -(\sigma_2 - \sigma_1) \int_0^1 \frac{z}{(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2} \exp\left(-\frac{z^2}{2(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2}\right) d\theta
\end{aligned}
\tag{2.51}
$$

then:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left| \left(G\left(\frac{z}{\sigma_2}\right) - G\left(\frac{z}{\sigma_1}\right) \right) \right| dz = |\sigma_2 - \sigma_1| \\
&\quad \times \int_{\mathbb{R}} \left| \int_0^1 \frac{z}{(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2} \exp\left(-\frac{z^2}{2(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2}\right) d\theta \right| dz \\
&\leq |\sigma_2 - \sigma_1| \int_0^1 \int_{\mathbb{R}} \frac{|z|}{(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2} \exp\left(-\frac{z^2}{2(\sigma_1 + \theta(\sigma_2 - \sigma_1))^2}\right) dz d\theta \\
&\leq |\sigma_2 - \sigma_1| \int_0^1 \int_{\mathbb{R}} \frac{|z|}{\sigma_1 + \theta(\sigma_2 - \sigma_1)} \exp\left(-\frac{1}{2}\left(\frac{z}{2(\sigma_1 + \theta(\sigma_2 - \sigma_1))}\right)^2\right) d\left(\frac{z}{\sigma_1 + \theta(\sigma_2 - \sigma_1)}\right) d\theta \\
&\leq |\sigma_2 - \sigma_1| \int_0^1 \int_{\mathbb{R}} |y| \exp\left(-\frac{y^2}{2}\right) dy d\theta \leq \frac{2}{\pi} |\sigma_2 - \sigma_1| \; .
\end{aligned}
\tag{2.52}
$$

We obtain that:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left| \left(G\left(\frac{z}{\Sigma_{xx}(t-s)}\right) - G\left(\frac{z}{\sigma\sqrt{t-s}}\right) \right) \right| dz \leq \left| \Sigma_{xx}(t-s) - \sigma\sqrt{t-s} \right| \\
&\leq \frac{\sigma^2(t-s) - \Sigma_{xx}^2(t-s)}{\Sigma_{xx}(t-s) + \sigma\sqrt{t-s}} \leq \sigma \frac{\frac{2}{\beta}(1 - e^{-\beta(t-s)}) - \frac{1}{2\beta}(1 - e^{-2\beta(t-s)})}{\sqrt{t-s}} \leq \frac{2\sigma}{\beta\sqrt{t-s}} \, .
\end{aligned}
\tag{2.53}
$$

Therefore, the L^1-norm of the difference between the two Gaussian with different variances in (2.40), is controlled by :

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^2} \mu(\cdot)(1 - e^{-\beta(t-s)})\rho(\cdot,\cdot,\cdot) \, dy dv ds \left(\partial_y g(0, \Sigma_{xx}^2(t-s), z) - \partial_y g(0, \sigma^2(t-s), z) \right) \Big|_{z = x - y - \frac{v}{\beta}(1 - e^{-\beta(t-s)})} \right\|_{L^1} \\
&\leq C_{\mu_0,\mu,\sigma,t} \frac{1}{\beta} \int_0^t \frac{1}{\sqrt{s}} \, ds \leq C_{\mu,\mu_0,\sigma,t} \frac{1}{\beta} \, ,
\end{aligned}
\tag{2.54}
$$

where the bound on the norm of the partial derivative of the density, $C_{\mu_0,\mu,\sigma,t}$, is also obtained from Lemma 2.7.

By this control and by the bound obtained in (2.48), we have that time integral term in the equation (2.34) is controlled as:

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^2} \mu(y) \left((1 - e^{-\beta(t-s)}) \partial_y M(t-s; y, v; x) - \partial_y \Gamma_{\mathrm{B}}(t-s; y; x) \right) \rho(s, y, v) \, dy dv ds \right\|_{L^1(\mathbb{R})} \\
&\leq C_{\mu_0,\mu,\sigma,t} \frac{\ln(\beta)}{\beta} \, .
\end{aligned}
\tag{2.55}
$$

Final steps

Considering the control on the initial term (2.39) and on the time integral term (2.55), we take the L^1-norm in the equation (2.34) to obtain:

$$\begin{aligned}\left\|\int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x)\right\|_{L^1(\mathbb{R})} &\leq \frac{1}{\beta}\left(\|v\partial_y\mu_0\|_{L^1(\mathbb{R}^2)} + 2\sigma^2\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}^2)}\right)\\ &+ C_{\mu,\mu_0,\sigma,t}\frac{\ln(\beta)}{\beta} + \|\mu\|_{L^\infty}\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\left\|\int_{\mathbb{R}}\rho(s,y,v)\,dv - p(s,y)\right\|_{L^1(\mathbb{R})} ds\end{aligned} \tag{2.56}$$

so by Gronwall's lemma:

$$\left\|\int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x)\right\|_{L^1(\mathbb{R})} \leq \left(\frac{1}{\beta}\left(\|v\partial_y\mu_0\|_{L^1(\mathbb{R}^2)} + 2\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}^2)}\right) + C_{\mu,\mu_0,\sigma,t}\frac{\ln(\beta)}{\beta}\right)e^{\|\mu\|^2_{L^\infty}\frac{2t}{\sigma^2}}. \tag{2.57}$$

So for $\beta > e$, we obtain that:

$$\left\|\int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x)\right\|_{L^1(\mathbb{R})} \leq \tilde{C}_{\mu_0,\mu,\sigma,t}\frac{\ln(\beta)}{\beta} \tag{2.58}$$

where:

$$\tilde{C}_{\mu_0,\mu,\sigma,t} = \left(\|v\partial_y\mu_0\|_{L^1(\mathbb{R}^2)} + 2\left\|\partial^2_{yy}\mu_0\right\|_{L^1(\mathbb{R}^2)} + C_{\mu,\mu_0,\sigma,t}\right)e^{\|\mu\|^2_{L^\infty}\frac{2t}{\sigma^2}}.$$

■

Corollary 2.6. *Assume ($H_{Forward}$) and consider $(x_s,u_s)_{s\geq 0}$ and $(Y_s)_{s\geq 0}$ solutions of SDE (1.6) and, respectively, (1.7), with initial condition μ_0 and $\mu_0^Y := \int_{\mathbb{R}}\mu_0(\cdot,dv)$. Then, for any $t >$ there exists a coefficient $C_{\mu_0,\mu,\beta,t}$ that does not depend on β such that for any measurable and bounded function f, we have that:*

$$\left|\mathbb{E}_{\mu_0}f(x_t) - \mathbb{E}_{\mu_0^Y}f(Y_t)\right| \leq C_{\mu_0,\mu,\beta,t}\|f\|_{L^\infty}\frac{\ln(\beta)}{\beta}. \tag{2.59}$$

Proof. Let $t > 0$ and f a measurable bounded function. Then by Theorem 2.5:

$$\begin{aligned}\left|\mathbb{E}_{\mu_0}f(x_t) - \mathbb{E}_{\mu_0^Y}f(Y_t)\right| &= \left|\int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}}\rho(t,x,u)\,du - p(t,x)\right)dx\right|\\ &\leq \|f\|_{L^\infty}\left\|\int_{\mathbb{R}}\rho(t,x,u)\,du - p(t,x)\right\|_{L^1(\mathbb{R})} \leq C_{\mu_0,\mu,\sigma,t}\|f\|_{L^\infty}\frac{\ln(\beta)}{\beta}.\end{aligned} \tag{2.60}$$

■

2.3 Bounds for the first derivative

Lemma 2.7. *Assume ($H_{Forward}$) and consider ρ the solution to the mild equation (2.28). Then, for any $T > 0$, we have that:*

(i) $\sup_{t\in[0,T]} \|\partial_x\rho(t,x,u)\|_{L^1(\mathbb{R}^2)} \leq C_{\mu_0,\mu,\sigma,T}$

(ii) $\sup_{t\in[0,T]} \|\partial_{xx}\rho(t,x,u)\|_{L^1(\mathbb{R}^2)} \leq C_{\mu_0,\mu,\sigma,T}$

(iii) $\sup_{t\in[0,T]}\left\|\int_{\mathbb{R}} u\rho(t,x,u)\,du\right\|_{L^1(\mathbb{R})} \le C_{\mu_0,\mu,\sigma,T}$

where $C_{\mu_0,\mu,\sigma,T}$ does not depend on β.

Proof. The proof for this lemma relies on differentiating the mild equation (2.28) and on performing several integration by parts in order to obtain a form where Gronwall's inequality can be applied.

***(i)* Norm of the first derivative**

We differentiate the mild equation (2.28) w.r.t. x:

$$\begin{aligned}\partial_x\rho(t,x,u) &= \int_{\mathbb{R}\times\mathbb{R}} \partial_x\Gamma_{\mathrm{OU}}(t;y,v;x,u)\mu_0(dy,dv)\\ &+\beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}} \partial_v\partial_x\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\rho(s,y,v)\mu(y)\,dydvds\,.\end{aligned} \tag{2.61}$$

Since $\partial_x\Gamma_{\mathrm{OU}}(t-s;y,v;x,u) = -\partial_y\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)$ and by performing an integration by parts (with null boundary terms as $\Gamma_{\mathrm{OU}}(\cdot;y,\cdot;\cdot,\cdot)\to 0$ at infinity and ρ has bounded tail):

$$\begin{aligned}\partial_x\rho(t,x,u) &= \int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\partial_y\mu_0(y,v)\,dydv\\ &+\beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}} \partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\partial_y\left(\rho(s,y,v)\mu(y)\right)\,dydvds\,,\end{aligned} \tag{2.62}$$

and taking the L^1-norm in x we obtain that:

$$\begin{aligned}\left\|\int_{\mathbb{R}} |\partial_x\rho(t,x,u)|\,du\right\|_{L^1(\mathbb{R})} &\le \left\|\int_{\mathbb{R}^2} M(t;y,v;x)\,|\partial_y\mu_0(y,v)|\,dydv\right\|_{L^1(\mathbb{R})}\\ &+\beta\int_0^t\int_{\mathbb{R}^2} |\partial_y\left(\rho(s,y,v)\mu(y)\right)|\,dydvds\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)|\,dxdu\,.\end{aligned} \tag{2.63}$$

We have two terms in this equation. The first corresponds to the initial condition and the second to the time integral. Fubini's theorem allows to integrate the initial condition in x firstly and since the function M is a probability density:

$$\left\|\int_{\mathbb{R}^2} M(t;y,v;x)\,|\partial_y\mu_0(y,v)|\,dydv\right\|_{L^1(\mathbb{R})} = \|\partial_y\mu_0(y,v)\|_{L^1(\mathbb{R}^2)}\,. \tag{2.64}$$

We recall that Γ_{OU} is the transition density of a solution to the SDE (1.6) in the no-drift, $\mu\equiv 0$, case. Since the process is Gaussian, this transition density is completely determined by the mean vector and the covariance matrix. These are presented in (A.5) and (A.4); There we introduce the functions:

$$\Sigma_{uu}\colon t\mapsto\sqrt{\beta\frac{\sigma^2}{2}(1-e^{-2\beta t})}$$

and $\rho\colon (0,+\infty)\mapsto(0,+\infty)$ such that

$$\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t) = \frac{\sigma^2}{2}(1-e^{-\beta t})^2\,.$$

We therefore have that:

$$\begin{aligned}\partial_v\Gamma_{\mathrm{OU}}(t;y,v;x,u) &= \Gamma_{\mathrm{OU}}(t;y,v;x,u)\left((x-y-\frac{v}{\beta}(1-e^{-\beta t}))\frac{(1-e^{-\beta t})\Sigma_{uu}(t)-\beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}^2(t)\Sigma_{uu}(t)}\right)\\ &+\Gamma_{\mathrm{OU}}(t;y,v;x,u)\left((u-ve^{-\beta t})\frac{\beta e^{-\beta t}\Sigma_{xx}(t)-(1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}^2(t)}\right)\end{aligned} \tag{2.65}$$

and integrating, we have that:

$$\int_{\mathbb{R}^2} |\partial_v \Gamma_{\text{OU}}(t; y, v; x, u)| \, dx du \leq C(\beta, \sigma, t) \,, \tag{2.66}$$

where:

$$\begin{aligned} C(\beta, \sigma, t) &:= \frac{1}{\sqrt{2\pi}} \frac{\left|(1-e^{-\beta t})\Sigma_{uu}(t) - \beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)\right|}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)} \\ &+ \frac{1}{\sqrt{2\pi}} \frac{\left|\beta e^{-\beta t}\Sigma_{xx}(t) - (1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)\right|}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)} \,. \end{aligned} \tag{2.67}$$

We can notice that the bound $C(\beta, \sigma, t)$ only depends on β, σ and t. Going back to (2.63) and developing the derivatives, we have that:

$$\begin{aligned} &\left\| \int_{\mathbb{R}} |\partial_x \rho(t, x, u)| \, du \right\|_{L^1(\mathbb{R})} \leq \|\partial_y \mu_0(y, v)\|_{L^1(\mathbb{R}^2)} + \beta \int_0^t C(\beta, \sigma, t-s) \left\| \int_{\mathbb{R}} |\partial_y \left(\rho(s, y, v)\mu(y)\right)| \, dv \right\|_{L_1(\mathbb{R})} ds \\ &\leq \|\partial_y \mu_0\|_{L^1(\mathbb{R}^2)} + \beta \|\partial_y \mu\|_{L^\infty} \int_0^t C(\beta, \sigma, t-s) \, ds + \beta \|\mu\|_{L^\infty} \int_0^t C(\beta, \sigma, t-s) \left\| \int_{\mathbb{R}} |\partial_y \rho(s, y, v)| \, dv \right\|_{L^1(\mathbb{R})} ds \,. \end{aligned} \tag{2.68}$$

By the Corollary 4.2, we have that there exists $C_\sigma > 0$ which depends on σ but not on β, such that

$$\beta C(\beta, \sigma, t-s) \leq C_\sigma \frac{1}{\sqrt{t-s}}$$

thus, for any $T \geq t$, the previous inequality becomes

$$\begin{aligned} &\left\| \int_{\mathbb{R}} |\partial_x \rho(t, x, u)| \, du \right\|_{L^1(\mathbb{R})} \leq \|\partial_y \mu_0(y, v)\|_{L^1(\mathbb{R}^2)} + \beta \int_0^t C(\beta, \sigma, t-s) \left\| \int_{\mathbb{R}} |\partial_y \left(\rho(s, y, v)\mu(y)\right)| \, dv \right\|_{L_1(\mathbb{R})} ds \\ &\leq \|\partial_y \mu_0\|_{L^1(\mathbb{R}^2)} + 2C_\sigma \|\partial_y \mu\|_{L^\infty} \sqrt{t} + \|\mu\|_{L^\infty} \int_0^t \frac{1}{\sqrt{t-s}} \left\| \int_{\mathbb{R}} |\partial_y \rho(s, y, v)| \, dv \right\|_{L^1(\mathbb{R})} ds \\ &\leq \|\partial_y \mu_0\|_{L^1(\mathbb{R}^2)} + 2C_\sigma \|\partial_y \mu\|_{L^\infty} \sqrt{T} + \|\mu\|_{L^\infty} \int_0^t \frac{C_\sigma}{\sqrt{t-s}} \left\| \int_{\mathbb{R}} |\partial_y \rho(s, y, v)| \, dv \right\|_{L^1(\mathbb{R})} ds \end{aligned} \tag{2.69}$$

and by applying Gronwall's inequality as presented in the Remark 4.4 in the Appendix, we have for any $T \geq t$:

$$\left\| \int_{\mathbb{R}} |\partial_x \rho(t, x, u)| \, du \right\|_{L^1(\mathbb{R})} \leq 2 \left(\|\partial_y \mu_0\|_{L^1(\mathbb{R}^2)} + 2C_\sigma \|\partial_y \mu\|_{L^\infty} \sqrt{T} \right) \exp\left(\pi \|\mu\|_{L^\infty}^2 C_\sigma^2 T \right) \,. \tag{2.70}$$

where C_σ depends only on σ. Taking the supremum on $[0, T]$ proves the bound for *(i)*.

(ii) **Norm of the second derivative** We differentiate the mild equation (2.28) w.r.t. x two times:

$$\begin{aligned} \partial_{xx} \rho(t, x, u) &= \int_{\mathbb{R} \times \mathbb{R}} \partial_{xx} \Gamma_{\text{OU}}(t; y, v; x, u) \mu_0(dy, dv) \\ &+ \beta \int_0^t \int_{\mathbb{R} \times \mathbb{R}} \partial_v \partial_{xx} \Gamma_{\text{OU}}(t-s; y, v; x, u) \rho(s, y, v) \mu(y) \, dy dv ds \end{aligned} \tag{2.71}$$

Since $\partial_{xx}\Gamma_{\mathrm{OU}}(t-s;y,v;x,u) = \partial_{yy}\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)$ and by performing an integration by parts:

$$\begin{aligned}\partial_{xx}\rho(t,x,u) &= \int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\partial_{yy}\mu_0(dy,dv) \\ &+ \beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}} \partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\partial_{yy}\left(\rho(s,y,v)\mu(y)\right)\,dydvds\end{aligned} \tag{2.72}$$

so again by taking the L^1- norm in x we have that:

$$\begin{aligned}\left\|\int_{\mathbb{R}} |\partial_{xx}\rho(t,x,u)|\,du\right\|_{L^1(\mathbb{R})} &\le \left\|\int_{\mathbb{R}^2} M(t;y,v;x)\,|\partial_{yy}\mu_0(dy,dv)|\right\|_{L^1(\mathbb{R})} \\ &+ \beta\int_0^t\int_{\mathbb{R}^2} |\partial_{yy}\mu(y)|\,\rho(s,y,v)\,dydvds\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)|\,dxdu \\ &+ 2\beta\int_0^t\int_{\mathbb{R}^2} |\partial_y\left(\rho(s,y,v)\mu(y)\right)|\,dydvds\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)|\,dxdu \\ &+ \beta\int_0^t\int_{\mathbb{R}^2} |\mu(y)\partial_{yy}\rho(s,y,v)|\,dydvds\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)|\,dxdu\end{aligned} \tag{2.73}$$

By utilising the bounds previously obtained for the norm of the first derivative of the density and Gronwall's inequality as in Remark 4.4, we can conclude that by taking the supremum on $[0,T]$ we have:

$$\sup_{t\in[0,T]}\left\|\int_{\mathbb{R}}|\partial_{xx}\rho(t,x,u)|\,du\right\|_{L^1(\mathbb{R})} \le C_{\mu_0,\mu,\sigma,T} \tag{2.74}$$

uniformly in β.

(iii) **Norm of the first moment of the velocity** We shall bound the first moment uniformly in β:

$$\begin{aligned}\int_{\mathbb{R}} u\rho(t,x,u)\,du &= \int_{\mathbb{R}\times\mathbb{R}}\left(\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t;y,v;x,u)\,du\right)\mu_0(dy,dv) \\ &+ \beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\partial_v\left(\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\,du\right)\rho(s,y,v)\mu(y)\,dydvds\,.\end{aligned} \tag{2.75}$$

We can see that this is a mild equation with kernel $\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}$. As previously we have a component that represents the initial condition and a time integral component which we shall bound but first which we further explicit the kernel:

$$\begin{aligned}\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t;y,v;x,u)\,du &= \frac{\rho(t)\Sigma_{uu}(t)}{\sqrt{2\pi}\Sigma_{xx}^2(t)}\left(x-y-\frac{v}{\beta}(1-e^{-\beta t})\right)\exp\left(-\frac{1}{2\Sigma_{xx}^2}\left(x-y-\frac{v}{\beta}(1-e^{-\beta t})\right)\right) \\ &+ ve^{-\beta t}M(t;y,v;x) = \rho(t)\Sigma_{uu}(t)\Sigma_{xx}(t)\partial_y M(t;y,v;x) + ve^{-\beta t}M(t;y,v;x)\end{aligned} \tag{2.76}$$

and by the definition of the marginal density M in (2.32), $\partial_v M(t;y,v;x,u) = \dfrac{1-e^{-\beta t}}{\beta}\partial_y M(t;y,v;x,u)$:

$$\begin{aligned}\partial_v\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t;y,v;x,u)\,du &= \rho(t)\Sigma_{uu}(t)\Sigma_{xx}(t)\frac{1-e^{-\beta t}}{\beta}\partial_{yy}M(t;y,v;x) + e^{-\beta t}M(t;y,v;x) \\ &+ \frac{v}{\beta}(1-e^{-\beta t})e^{-\beta t}\partial_y M(t;y,v;x)\,.\end{aligned} \tag{2.77}$$

For the term that contains the initial condition:

$$
\begin{aligned}
&\left\|\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t;y,v;x,u)\,du\right)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})} \le \rho(t)\Sigma_{uu}(t)\Sigma_{xx}(t)\left\|\int_{\mathbb{R}^2}\partial_y M(t;y,v;x)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})}\\
&\quad + e^{-\beta t}\left\|\int_{\mathbb{R}^2} vM(t;y,v;x)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})}\\
&\le \sigma^2\left\|\int_{\mathbb{R}^2} M(t;y,v;x)\partial_y\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})} + \left\|\int_{\mathbb{R}^2} vM(t;y,v;x)\mu_0(dy,dv)\right\|_{L^1(\mathbb{R})}\\
&\le \sigma^2\left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \left\|v\mu_0\right\|_{L^1(\mathbb{R}^2)}\,.
\end{aligned}
\tag{2.78}
$$

We now consider the time integral term from the r.h.s. of equation (2.75), which we write explicitly using the expression for the kernel in (2.76):

$$
\begin{aligned}
&\beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\partial_v\left(\int_{\mathbb{R}} u\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\,du\right)\rho(s,y,v)\mu(y)\,dydvds\\
&=\int_0^t \rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)})\int_{\mathbb{R}\times\mathbb{R}}\partial_{yy}M(t-s;y,v;x)\rho(s,y,v)\mu(y)\,dydvds\\
&\quad+\beta\int_0^t e^{-\beta(t-s)}\int_{\mathbb{R}\times\mathbb{R}} M(t-s;y,v;x)\rho(s,y,v)\mu(y)\,dydvds\\
&\quad+\int_0^t(1-e^{-\beta(t-s)})e^{-\beta(t-s)}\int_{\mathbb{R}\times\mathbb{R}}\partial_y M(t-s;y,v;x)v\rho(s,y,v)\mu(y)\,dydvds\,.
\end{aligned}
\tag{2.79}
$$

The three terms of this equality are all analysed separately.
The first term of (2.79), is transformed by transferring the derivatives from the density M to the density ρ and applying the estimates on the first and second derivative of the density, already proven before:

$$
\begin{aligned}
&\left\|\int_0^t \rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)})\int_{\mathbb{R}\times\mathbb{R}}\partial_{yy}M(t-s;y,v;x)\rho(s,y,v)\mu(y)\,dydvds\right\|_{L^1(\mathbb{R})}\\
&\le\sigma^2 t\left(\left\|\partial_{yy}\mu\right\|_{L^\infty(\mathbb{R}^2)}+2\left\|\partial_y\mu\right\|_{L^\infty(\mathbb{R}^2)}\left\|\partial_y\rho(t,\cdot,\cdot)\right\|_{L^1(\mathbb{R}^2)}+\left\|\mu\right\|_{L^\infty(\mathbb{R}^2)}\left\|\partial_{yy}\rho(t,\cdot,\cdot)\right\|_{L^1(\mathbb{R}^2)}\right)\,.
\end{aligned}
\tag{2.80}
$$

Since for all $s\le t$ according to (A.4), $\rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)})\le\sigma^2$, and by the bounds on the norm of the first derivative in *(i)* and the second derivative *(ii)*, we have that this first term is bounded uniformly in β.
Second term of (2.79) is:

$$
\left\|\beta\int_0^t e^{-\beta(t-s)}\int_{\mathbb{R}\times\mathbb{R}} M(t-s;y,v;x)\rho(s,y,v)\mu(y)\,dydvds\right\|_{L^1(\mathbb{R})}\le\left\|\mu\right\|_{L^\infty(\mathbb{R})}\beta\int_0^t e^{-\beta(t-s)}\,ds\le\left\|\mu\right\|_{L^\infty(\mathbb{R})}\,.
\tag{2.81}
$$

The third term of (2.79) is:

$$
\begin{aligned}
&\left\|\int_0^t(1-e^{-\beta(t-s)})e^{-\beta(t-s)}\int_{\mathbb{R}\times\mathbb{R}}\partial_y M(t-s;y,v;x)v\rho(s,y,v)\mu(y)\,dydvds\right\|_{L^1(\mathbb{R})}\\
&\le\sqrt{\frac{2}{\pi}}\left\|\mu\right\|_{L^\infty(\mathbb{R})}\int_0^t(1-e^{-\beta(t-s)})e^{-\beta(t-s)}\frac{1}{\Sigma_{xx}(t-s)}\left\|\int_{\mathbb{R}}\left|v\rho(s,y,v)\right|\,dv\right\|_{L^1(\mathbb{R})}\,ds\\
&\le\left\|\mu\right\|_{L^\infty(\mathbb{R})}C_{\mu_0,\mu,\sigma}\sqrt{\beta}\int_0^t\frac{(1-e^{-\beta s})e^{-\beta s}}{\Sigma_{xx}(s)}\,ds\le\left\|\mu\right\|_{L^\infty(\mathbb{R})}C_{\mu_0,\mu,\sigma}C_\sigma\,.
\end{aligned}
\tag{2.82}
$$

By the bound in (A.22), we have that this term is bounded uniformly in β. We also used the control of the first moment in absolute value from Lemma 2.8-(i). Thus the three terms of the equality (2.79) are bounded uniformly in β. Together with the similar result for the initial condition (2.78), we obtain that for any $T > 0$

$$\sup_{t\in[0,T]} \left\| \int_{\mathbb{R}} u\rho(t,x,u)\, du \right\|_{L^1(\mathbb{R})} \leq C_{\mu_0,\mu,\sigma,T} \tag{2.83}$$

where the bounding term does not depend on β.

■

Lemma 2.8. *Assume ($H_{Forward}$) and consider ρ the solution to the mild equation* (2.28)*, then we have the following controls for any $T > 0$:*

(i) $\|u\rho(t,x,u)\|_{L^1(\mathbb{R}^2)} < C_{\mu_0,\mu,\sigma}\sqrt{\beta}$

(ii) $\left\|u^2\rho(t,x,u)\right\|_{L^1(\mathbb{R}^2)} < C_{\mu_0,\mu,\sigma}\beta$

(iii) $\sup_{t\in[0,T]} \left\|u^2\partial_x\rho(t,x,u)\right\|_{L^1(\mathbb{R}^2)} < C_{\mu_0,\mu,\sigma,T}\beta\,.$

Proof.

(i) **Norm of the velocity** Since the integrated term is positive and by (A.4) $\Sigma_{uu}(t) \leq \sigma\sqrt{\beta}$, we have that:

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \int_{\mathbb{R}} |u|\rho(t,x,u)\, du \right| dx = \int_{\mathbb{R}}\int_{\mathbb{R}} |u|\rho(t,x,u)\, du\, dx = \int_{\mathbb{R}} |u| \int_{\mathbb{R}} \rho(t,x,u)\, dxdu \\
&= \int_{\mathbb{R}\times\mathbb{R}} |u| \left(\int_{\mathbb{R}} \Gamma_{\mathrm{OU}}(t;\cdot,v;x,u)\, dx \right) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&\quad + \beta \int_0^t \int_{\mathbb{R}} \partial_v \left(\int_{\mathbb{R}} |u| \int_{\mathbb{R}} \Gamma_{\mathrm{OU}}(t-s;\cdot,v;x,u)\, dx\, du \right) \int_{\mathbb{R}} \rho(s,y,v)\mu(y)\, dydvds \\
&= \int_{\mathbb{R}} \left(\sqrt{\frac{2}{\pi}}\Sigma_{uu}(t) \exp\left(-\frac{v^2 e^{-2\beta t}}{2\Sigma_{uu}^2(t)} \right) + e^{-\beta t} v \,\mathrm{erf}\left(\frac{v e^{-\beta t}}{\sqrt{2}\Sigma_{uu}(t)} \right) \right) \int_{\mathbb{R}} \mu_0(dy,dv) \\
&\quad + \beta \int_0^t \int_{\mathbb{R}} e^{-\beta(t-s)}\,\mathrm{erf}\left(\frac{v e^{-\beta(t-s)}}{\sqrt{2}\Sigma_{uu}(t-s)} \right) \int_{\mathbb{R}} \rho(s,y,v)\mu(y)\, dydvds \\
&\leq \sigma\sqrt{\beta} + \int_{\mathbb{R}\times\mathbb{R}} |v|\mu_0(dy,dv) + \|\mu\|_{L^\infty(\mathbb{R})}\,.
\end{aligned} \tag{2.84}$$

Thus for $C^{(i)}(\mu_0,\mu,\sigma) := 2\max\left\{ \sigma, \int_{\mathbb{R}\times\mathbb{R}} |v|\mu_0(dy,dv) + \|\mu\|_{L^\infty(\mathbb{R})} \right\}$:

$$\|u\rho(t,x,u)\|_{L^1(\mathbb{R}^2)} \leq C^{(i)}(\mu_0,\mu,\sigma)\sqrt{\beta}\,. \tag{2.85}$$

(ii) **Norm of the second moment of the velocity**

Again since the integrated terms are positive:

$$
\begin{aligned}
&\int_{\mathbb{R}}\left|\int_{\mathbb{R}} u^2\rho(t,x,u)\,du\right|\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} u^2\rho(t,x,u)\,du\,dx = \int_{\mathbb{R}} u^2\int_{\mathbb{R}}\rho(t,x,u)\,dxdu\\
&= \int_{\mathbb{R}\times\mathbb{R}} u^2\left(\int_{\mathbb{R}}\Gamma_{\mathrm{OU}}(t;\cdot,v;x,u)\,dx\right)\int_{\mathbb{R}}\mu_0(dx,dv)\\
&\quad+\beta\int_0^t\int_{\mathbb{R}}\partial_v\left(\int_{\mathbb{R}} u^2\int_{\mathbb{R}}\Gamma_{\mathrm{OU}}(t-s;\cdot,v;x,u)\,dx\,du\right)\int_{\mathbb{R}}\rho(s,y,v)\mu(y)\,dydvds\\
&= \int_{\mathbb{R}}\left(\Sigma_{uu}^2(t)+v^2e^{-2\beta t}\right)\int_{\mathbb{R}}\mu_0(dy,dv)\\
&\quad+\int_0^t\beta\sigma^2(1-e^{-2\beta t})\int_{\mathbb{R}} e^{-\beta(t-s)}\int_{\mathbb{R}} v\rho(s,y,v)\mu(y)\,dydvds\\
&\le \sigma^2\beta+\left\|v^2\mu_0\right\|_{L^1(\mathbb{R})}+\sigma^2\left\|\mu\right\|_{L^\infty(\mathbb{R})}C^{(i)}(\mu_0,\mu,\sigma)\sqrt{\beta}
\end{aligned}
\tag{2.86}
$$

So by taking $C^{(ii)}=2\max\left\{\sigma^2(1+\|\mu\|_{L^\infty(\mathbb{R})}C^{(i)}(\mu_0,\mu,\sigma)),\left\|v^2\mu_0\right\|_{L^1(\mathbb{R}^2)}\right\}$, for $\beta\ge 1$

$$
\left\|u^2\rho\right\|_{L^1(\mathbb{R}^2)}\le C^{(ii)}(\mu_0,\mu,\sigma)\beta\,. \tag{2.87}
$$

(iii) **Second moment and derivative**

We differentiate the mild equation (2.28) and by using $\partial_x\Gamma_{\mathrm{OU}}(t;y,v;x,u)=-\partial_y\Gamma_{\mathrm{OU}}(t;y,v;x,u)$ we perform an i.b.p to obtain:

$$
\begin{aligned}
\partial_x\rho(t,x,u) &= \int_{\mathbb{R}\times\mathbb{R}}\Gamma_{\mathrm{OU}}(t;y,v;x,u)\partial_y\mu_0(y,v)\,dydv\\
&\quad+\beta\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\partial_y\left(\rho(s,y,v)\mu(y)\right)\,dydvds\,.
\end{aligned}
\tag{2.88}
$$

For the initial term we bound and take the L^1 norm:

$$
\begin{aligned}
&\int_{\mathbb{R}^4} u^2\Gamma_{\mathrm{OU}}(t;y,v;x,u)\left|\partial_y\mu_0(dy,dv)\right|dxdu = \int_{\mathbb{R}^2}\left(v^2e^{-2\beta t}+\frac{\sigma^2}{2}\beta(1-e^{-\beta t})\right)\left|\partial_y\mu_0(dy,dv)\right|\\
&\le \left\|v^2\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)}+\sigma^2\beta\left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)}\,.
\end{aligned}
\tag{2.89}
$$

Now we have:

$$
\begin{aligned}
\partial_v\Gamma_{\mathrm{OU}}(t;y,v;x,u) &= \Gamma_{\mathrm{OU}}(t;y,v;x,u)\left((x-y-\frac{v}{\beta}(1-e^{-\beta t}))\frac{(1-e^{-\beta t})\Sigma_{uu}(t)-\beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}^2(t)\Sigma_{uu}(t)}\right)\\
&\quad+\Gamma_{\mathrm{OU}}(t;y,v;x,u)\left((u-v^{-\beta t})\frac{\beta e^{-\beta t}\Sigma_{xx}(t)-(1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}^2(t)}\right)\,.
\end{aligned}
\tag{2.90}
$$

We denote by

$$
I_{\mathrm{I}}(\beta,\sigma,t) := \frac{(1-e^{-\beta t})\Sigma_{uu}(t)-\beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}^2(t)\Sigma_{uu}(t)}
$$

and by

$$
I_{\mathrm{II}}(\beta,\sigma,t) := \frac{\beta e^{-\beta t}\Sigma_{xx}(t)-(1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}^2(t)}
$$

Thus, we have that:

$$
\begin{aligned}
&\int_{\mathbb{R}^2} u^2 \left|\partial_v \Gamma_{\mathrm{OU}}(t;y,v;x,u)\right| \, dxdu \leq |I_{\mathrm{I}}(\beta,\sigma,t)| \int_{\mathbb{R}^2} u^2 \Gamma_{\mathrm{OU}}(t;y,v;x,u) \left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right| \, dxdu \\
&\quad + |I_{\mathrm{II}}(\beta,\sigma,t)| \int_{\mathbb{R}^2} u^2 \Gamma_{\mathrm{OU}}(t;y,v;x,u) \left| u - ve^{-\beta t} \right| \, dxdu
\end{aligned}
\tag{2.91}
$$

and we calculate each term of (2.91). We integrate firstly in u the first term:

$$
\begin{aligned}
&\int_{\mathbb{R}} u^2 \Gamma_{\mathrm{OU}}(t;y,v;x,u) \, du \\
&= \frac{e^{-\frac{1}{2\Sigma_{xx}^2(t)}(x-y-\frac{v}{\beta}(1-e^{-\beta t}))^2}}{\sqrt{2\pi}\Sigma_{xx}^3(t)} \Bigg(v^2 e^{-2\beta t} \Sigma_{xx}^2(t) + 2(x-y-\frac{v}{\beta}(1-e^{-\beta t})) v e^{-\beta t} \rho(t) \Sigma_{xx}(t) \Sigma_{uu}(t) \\
&\quad + (1-\rho(t)^2) \Sigma_{xx}^2(t) \Sigma_{uu}^2(t) + \rho^2(t) \Sigma_{uu}^2(t) (x-y-\frac{v}{\beta}(1-e^{-\beta t}))^2 \Bigg) .
\end{aligned}
\tag{2.92}
$$

We multiply each of these terms by $\left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right|$ and integrate in x to obtain:

$$
\int_{\mathbb{R}} \left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right| \frac{e^{-\frac{1}{2\Sigma_{xx}^2(t)}(x-y-\frac{v}{\beta}(1-e^{-\beta t}))^2}}{\sqrt{2\pi}\Sigma_{xx}^3(t)} v^2 e^{-2\beta t} \Sigma_{xx}^2(t) \, dx = \sqrt{\frac{2}{\pi}} v^2 e^{-2\beta t} \Sigma_{xx}(t) .
\tag{2.93}
$$

The second term is 0 since we integrate an odd function, after performing a change of variable on x. The third term gives:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right| \frac{e^{-\frac{1}{2\Sigma_{xx}^2(t)}(x-y-\frac{v}{\beta}(1-e^{-\beta t}))^2}}{\sqrt{2\pi}\Sigma_{xx}^3(t)} (1-\rho(t)^2) \Sigma_{xx}^2(t) \Sigma_{uu}^2(t) \, dx \\
&= (1-\rho(t)^2) \Sigma_{xx}(t) \Sigma_{uu}^2(t)
\end{aligned}
\tag{2.94}
$$

while the third term gives that:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right|^3 \frac{e^{-\frac{1}{2\Sigma_{xx}^2(t)}(x-y-\frac{v}{\beta}(1-e^{-\beta t}))^2}}{\sqrt{2\pi}\Sigma_{xx}^3(t)} \rho^2(t) \Sigma_{uu}^2(t) \, dx \\
&= 2\sqrt{\frac{2}{\pi}} \rho^2(t) \Sigma_{xx}(t) \Sigma_{uu}^2(t)
\end{aligned}
\tag{2.95}
$$

and we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} u^2 \Gamma_{\mathrm{OU}}(t;y,v;x,u) \left| x-y-\frac{v}{\beta}(1-e^{-\beta t}) \right| \, dxdu = \Sigma_{xx}(t) \left(\sqrt{\frac{2}{\pi}} v^2 e^{-2\beta t} + \left(1 + \left(2\sqrt{\frac{2}{\pi}} - 1 \right) \rho^2(t) \right) \Sigma_{uu}^2(t) \right) \\
&\leq \Sigma_{xx}(t) \left(v^2 e^{-2\beta t} + 2\Sigma_{uu}^2(t) \right) .
\end{aligned}
\tag{2.96}
$$

For the second r.h.s. term of (2.91), we integrate in x to obtain the marginal of the density Γ_{OU} in u:

$$\begin{aligned}
&\int_{\mathbb{R}} u^2 \left|u - ve^{-\beta t}\right| \frac{e^{-\frac{1}{2\Sigma_{uu}(t)^2}(u-ve^{-\beta t})^2}}{\sqrt{2\pi}\Sigma_{uu}(t)}\,du \le \int_{\mathbb{R}} 2\left|u - ve^{-\beta t}\right|^3 \frac{e^{-\frac{1}{2\Sigma_{uu}(t)^2}(u-ve^{-\beta t})^2}}{\sqrt{2\pi}\Sigma_{uu}(t)}\,du \\
&\quad + 2\int_{\mathbb{R}} v^2 e^{-2\beta t}\left|u - ve^{-\beta t}\right| \frac{e^{-\frac{1}{2\Sigma_{uu}(t)^2}(u-ve^{-\beta t})^2}}{\sqrt{2\pi}\Sigma_{uu}(t)}\,du \\
&\le 4\sqrt{\frac{2}{\pi}}\Sigma_{uu}^3(t) + 2\sqrt{\frac{2}{\pi}} v^2 e^{-2\beta t}\Sigma_{uu}(t)
\end{aligned} \tag{2.97}$$

We have that we can further bound (2.91):

$$\begin{aligned}
&\int_{\mathbb{R}^2} u^2 \left|\partial_v \Gamma_{\text{OU}}(t;y,v;x,u)\right|\,dxdu \le \left|I_{\text{I}}(\beta,\sigma,t)\right| \Sigma_{xx}(t)\left(v^2 e^{-2\beta t} + 2\Sigma_{uu}^2(t)\right) \\
&\quad + \sqrt{\frac{2}{\pi}}\Sigma_{uu}(t)\left|I_{\text{II}}(\beta,\sigma,t)\right|\left(4\Sigma_{uu}^2(t) + 2v^2e^{-2\beta t}\right) \\
&\le \left|\frac{(1-e^{-\beta t})\Sigma_{uu}(t) - \beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)}\right|\left(v^2e^{-2\beta t} + 2\Sigma_{uu}^2(t)\right) \\
&\quad + \frac{\left|\beta e^{-\beta t}\Sigma_{xx}(t) - (1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)\right|}{\beta(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)}(2v^2e^{-2\beta t} + 4\Sigma_{uu}^2(t))
\end{aligned} \tag{2.98}$$

thus:

$$\begin{aligned}
&\left\|\int_{\mathbb{R}} u^2 \left|\partial_x \rho(t,x,u)\right|\,du\right\|_{L^1(\mathbb{R})} \le \left\|v^2\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \sigma^2\beta\left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} \\
&\quad + \int_0^t \frac{\left|(1-e^{-\beta(t-s)})\Sigma_{uu}(t-s) - \beta e^{-\beta(t-s)}\rho(t-s)\Sigma_{xx}(t-s)\right|}{(1-\rho^2(t-s))\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\int_{\mathbb{R}^2}\left(v^2e^{-2\beta(t-s)} + 2\Sigma_{uu}^2(t-s)\right)\left|\partial_y(\rho\mu)\right|\,dsdydv \\
&\quad + 2\int_0^t \frac{\left|\beta e^{-\beta(t-s)}\Sigma_{xx}(t-s) - (1-e^{-\beta(t-s)})\rho(t-s)\Sigma_{uu}(t-s)\right|}{(1-\rho^2(t-s))\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\int_{\mathbb{R}^2}(v^2e^{-2\beta(t-s)} + 2\Sigma_{uu}^2(t-s))\left|\partial_y(\rho\mu)\right|\,dsdydv \\
&\le \left\|v^2\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \sigma^2\beta\left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} \\
&\quad + 8\sigma^2\beta\int_0^t\left(\frac{\left|(1-e^{-\beta(t-s)})\Sigma_{uu}(t-s) - \beta e^{-\beta(t-s)}\rho(t-s)\Sigma_{xx}(t-s)\right|}{(1-\rho^2)\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\right. \\
&\qquad \left. + \frac{\left|\beta e^{-\beta(t-s)}\Sigma_{xx}(t-s) - (1-e^{-\beta(t-s)})\rho(t-s)\Sigma_{uu}(t-s)\right|}{(1-\rho^2(t-s))\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\right)\left(\left\|\partial_y\mu\right\|_{L^\infty}\left\|\mu_0\right\|_{L^1(\mathbb{R}^2)} + \left\|\mu\right\|_{L^\infty}\left\|\partial_y\rho\right\|_{L^1(\mathbb{R}^2)}\right)ds \\
&\quad + 8\int_0^t\left(\frac{\left|(1-e^{-\beta(t-s)})\Sigma_{uu}(t-s) - \beta e^{-\beta(t-s)}\rho(t-s)\Sigma_{xx}(t-s)\right|}{(1-\rho^2(t-s))\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\right. \\
&\qquad \left. + \frac{\left|\beta e^{-\beta(t-s)}\Sigma_{xx}(t-s) - (1-e^{-\beta(t-s)})\rho(t-s)\Sigma_{uu}(t-s)\right|}{(1-\rho^2(t-s))\Sigma_{xx}(t-s)\Sigma_{uu}(t-s)}\right)\left(\left\|\partial_y\mu\right\|_{L^\infty}\left\|v^2\rho\right\|_{L^1(\mathbb{R}^2)} + \left\|\mu\right\|_{L^\infty}\left\|v^2\partial_y\rho\right\|_{L^1(\mathbb{R}^2)}\right)ds\,.
\end{aligned} \tag{2.99}$$

By Corollary 4.2, we have that there exists $C_\sigma > 0$ such that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}} u^2 \left|\partial_x \rho(t,x,u)\right|\,du\right\|_{L^1(\mathbb{R})} \le \left\|v^2\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \sigma^2\beta\left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} \\
&\quad + 2C_\sigma\int_0^t \frac{1}{\sqrt{t-s}}\int_{\mathbb{R}^2} v^2e^{-2\beta(t-s)}\left|\partial_y(\rho\mu)\right|\,dsdydv + 4C_\sigma\int_0^t\frac{1}{\sqrt{t-s}}\int_{\mathbb{R}^2}\Sigma_{uu}^2(t-s)\left|\partial_y(\rho\mu)\right|\,dsdydv\,.
\end{aligned} \tag{2.100}$$

We consider firstly, for any $T \geq t$

$$\int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^2} \Sigma^2_{uu}(t-s) \left|\partial_y (\rho\mu)\right| \, dsdydv \leq 2\beta \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^2} \left(|\mu\partial_y\rho| + \rho|\partial_y\mu|\right) \, dsdydv$$
$$\leq 4\beta C_{\mu_0,\mu,\sigma,T} \left\|\mu\right\|_{L^\infty} \sqrt{t} + 4\beta \left\|\partial_y\mu\right\|_{L^\infty} \sqrt{t} \leq \hat{C}_{\mu_0,\mu,\sigma,T}\beta$$

where the bounds on the norm of the derivative of the density from Lemma 2.7 have been used. We also consider

$$\int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^2} v^2 e^{-2\beta(t-s)} \left|\partial_y (\rho\mu)\right| \, dsdydv \leq \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^2} v^2 \left(|\mu\partial_y\rho| + \rho|\partial_y\mu|\right) \, dsdydv$$
$$\leq \left\|\mu\right\|_{L^\infty} \int_0^t \frac{1}{\sqrt{t-s}} \left\|v^2\partial_y\rho(s,\cdot,\cdot)\right\|_{L^1(\mathbb{R}^2)} \, ds + \left\|\partial_y\mu\right\|_{L^\infty} C_{\mu_0,\mu,\sigma}\sqrt{T}\beta$$

where the control of the second moment *(ii)* to bound the second term was used. Thus

$$\left\| \int_{\mathbb{R}} u^2 \left|\partial_x\rho(t,x,u)\right| \, du \right\|_{L^1(\mathbb{R})} \leq \left\|v^2\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \sigma^2\beta \left\|\partial_y\mu_0\right\|_{L^1(\mathbb{R}^2)} + \hat{C}_{\mu_0,\mu,\sigma,T}\beta + \left\|\partial_y\mu\right\|_{L^\infty} C_{\mu_0,\mu,\sigma}\sqrt{T}\beta$$
$$+ 2C_\sigma \left\|\mu\right\|_{L^\infty} \int_0^t \frac{1}{\sqrt{t-s}} \left\|v^2\partial_y\rho(s,\cdot,\cdot)\right\|_{L^1(\mathbb{R}^2)} \, ds \tag{2.101}$$

So by Gronwall's inequality, considering the bound of Corollary 4.2, we have that for any $T > 0$:

$$\sup_{t\in[0,T]} \left\|u^2\partial_x\rho\right\|_{L^1(\mathbb{R}^2)} \leq C_{\mu_0,\mu,\sigma,T}\beta \,. \tag{2.102}$$

■

3 Application to reflection

The case of reflection involves extending either the process or the mild equation to the whole domain. We introduce two sets of hypotheses that are useful for extending the drift to a differentiable odd or even function on the whole domain.

3.1 Bounding the error for odd drifts

We denote by $\mathcal{D} = [0,+\infty)$. Let $\mu\colon \mathcal{D} \mapsto \mathbb{R}$ be a bounded drift, consider the function $\widetilde{\mu}\colon x \in \mathbb{R} \mapsto \operatorname{sign}(x)\mu(|x|)$, and define the process $(\widetilde{y}_t, \widetilde{v}_t)_{t\geq 0}$ solution of:

$$\begin{cases} \widetilde{y}_t = x_0 + \displaystyle\int_0^t \widetilde{v}_s \, ds \\ \widetilde{v}_t = u_0 - \beta \displaystyle\int_0^t \widetilde{v}_s \, ds + \beta \int_0^t \widetilde{\mu}(\widetilde{y}_s) \, ds + \beta\sigma\widetilde{W}_t \,. \end{cases} \tag{3.1}$$

Hypotheses 3.1
We introduce the following set of hypotheses ($H_{\text{Reflected Odd}}$):

($H_{\text{Reflected Odd}}$)-*(i)* $\mu_0\colon (x,u) \in (\mathcal{D}\times\mathbb{R}) \mapsto [0,1]$ *is a probability measure with density that we also denote as* μ_0*, such that* $\partial_x\mu_0, \partial_{xx}\mu_0 \in L^1(\mathcal{D}\times\mathbb{R}) \cap L^\infty(\mathcal{D}\times\mathbb{R})$*, the integrals* $\int_{\mathcal{D}\times\mathbb{R}}(|u| + u^2)\mu_0(dx,du)$ *and* $\int_{\mathcal{D}\times\mathbb{R}}(|u| + u^2)|\partial_x\mu_0|(x,u)\,dxdu$ *are bounded. We also assume that* μ_0 *is zero on a neighbourhood of* $x = 0$ *and vanishes as infinity.*

($H_{\textit{Reflected Odd}}$)-$(ii)$$\mu\colon \mathcal{D} \mapsto \mathbb{R}$ is bounded, $\mu(0) = 0$ and $\mu', \mu'' \in L^\infty((0, +\infty))$.

We can extend, under the hypothesis ($H_{\textit{Reflected Odd}}$)-$(i)$, μ_0 on the whole domain $\mathbb{R} \times \mathbb{R}$, by making it equal to zero on $\mathcal{D}^c \times \mathbb{R}$ ($\mathcal{D}^c$ being the complement of $\mathcal{D}$). We denote this extension also as μ_0 and it is easy to see that it also verifies hypothesis ($H_{\textit{Forward}}$)-(i).

Under ($H_{\textit{Reflected Odd}}$)-$(ii)$, we have that $\widetilde{\mu}$ is such that the hypothesis ($H_{\textit{Forward}}$)-(ii) is verified for the process $(\widetilde{y}_t, \widetilde{v}_t)_{t\geq 0}$ so we can apply Corollary 2.6. If we introduce the process $(Y^f_t)_{t\geq 0}$ such that

$$Y^f_t = x_0 + \int_0^t \widetilde{\mu}(Y^f_s)\, ds + \sigma \widetilde{W}_t\,.$$

Let $\mu_0^Y = \int_{\mathbb{R}} \mu_0(\cdot, v)\, dv$, then for any g measurable bounded function:

$$\left| \mathbb{E}_{\mu_0} g(\widetilde{y}_t) - \mathbb{E}_{\mu_0^Y} g(Y^f_t) \right| \leq C_{\mu_0,\mu,\sigma,t} \, \|g\|_{L^\infty} \frac{\ln(\beta)}{\beta}\,. \tag{3.2}$$

We now assume that the position process is confined in the domain $\mathcal{D}$ and we have the following SDE obtained from the equations (1.1) from Chapter 1

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s\, ds, \\ u_t = u_0 - \beta \displaystyle\int_0^t u_s\, ds + \beta \int_0^t \mu(x_s)\, ds + \beta\sigma W_t - \sum_{0<s\leq t} 2u_{s^-} \mathbb{1}_{\{x_s = 0\}}\,. \end{cases} \tag{3.3}$$

By several results from [Bossy and Jabir, 2011], we have that the process defined as $(\mathfrak{X}_t, \mathfrak{U}_t)_{t\geq 0} = (|\widetilde{y}_t|, \operatorname{sign}(\widetilde{y}_t)\widetilde{v}_t)_{t\geq 0}$ is the weak solution of (3.3).

We also consider the process $(Y_t)_{t\geq 0}$ defined as $(Y_t)_{t\geq 0} = (|Y^f_t|)_{t\geq 0}$. Then, by Tanaka's formula:

$$\begin{aligned} Y_t &= x_0 + \int_0^t \operatorname{sign}(Y^f_s)\, dY^f_s + L^{Y^f}_t = x_0 + \int_0^t \operatorname{sign}(Y^f_s) \widetilde{\mu}(Y^f_s)\, ds + \sigma \int_0^t \operatorname{sign}(Y^f_s)\, d\widetilde{W}_s + L^{Y^f}_t \\ &= x_0 + \int_0^t \mu\left(|Y^f_s|\right)\, ds + \sigma \int_0^t \operatorname{sign}(Y^f_s)\, d\widetilde{W}_s + L^{Y^f}_t \\ &= x_0 + \int_0^t \mu\left(Y_t\right)\, ds + \sigma \int_0^t \operatorname{sign}(Y^f_s)\, d\widetilde{W}_s + L^{Y^f}_t \end{aligned} \tag{3.4}$$

where $(L^{Y^f}_t)_{t\geq 0}$ is the local time of $(Y^f_t)_{t\geq 0}$ at 0 and we have that:

$$L^{Y^f}_t = \frac{1}{2} L^Y_t$$

where $(L^Y_t)_{t\geq 0}$ is the local time of $(Y_t)_{t\geq 0}$ at 0. Since $\left\langle \int_0^\cdot \operatorname{sign}(Y^f_s)\, d\widetilde{W}_s \right\rangle_t = t$, then by Lévy's representation theorem, we have that $\left(\int_0^t \operatorname{sign}(Y^f_s)\, d\widetilde{W}_s \right)_{t\geq 0}$ is a Brownian motion. We can conclude that the process $(Y_t)_{t\geq 0} = (|Y^f_t|)_{t\geq 0}$ is a weak solution of the SDE:

$$\begin{cases} Y_t = x_0 + \displaystyle\int_0^t \mu(Y_s)\, ds + \sigma W_t + \frac{1}{2} L^Y_t \\ L^Y_t = \displaystyle\int_0^t \mathbb{1}_{\{0\}}(Y_s)\, dL^Y_s\,. \end{cases} \tag{3.5}$$

Let g be a measurable, bounded function, $(x_t, u_t)_{t\geq 0}$ a weak solution to the specularly reflected SDE (3.3), $(Y_t)_{t\geq 0}$ a weak solution to the confined diffusion (3.5), then by (3.2):

$$\left|\mathbb{E}_{\mu_0} g(x_t) - \mathbb{E}_{\mu_0^Y} g(Y_t)\right| = \left|\mathbb{E}_{\mu_0} g(|\widetilde{y}_t|) - \mathbb{E}_{\mu_0^Y} g(|Y_t^f|)\right| \leq C_{\mu_0,\mu,\sigma,t} \, \|g\|_{L^\infty} \frac{\ln(\beta)}{\beta} \,. \tag{3.6}$$

3.2 Bounding the error for even drifts

We now consider the mild equation verified by the density ρ of the process $(x_t, u_t)_{t\geq 0}$ solution of the SDE (3.3), obtained in [Bossy and Jabir, 2011]. For any $(t, x, u) \in (0, T] \times \mathcal{D} \times \mathbb{R}$:

$$\rho(t, x, u) = \int_{\mathcal{D}\times\mathbb{R}} g_c(t; y, v; x, u)\mu_0(y, v)\, dydv + \beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \partial_v g_c(t-s; y, v; x, u)\mu(y)\rho(s, y, v)\, dsdydv \tag{3.7}$$

where for any $(t, y, v, x, u) \in \mathbb{R}^+ \times (0, \infty) \times \mathbb{R} \times \mathcal{D} \times \mathbb{R}$

$$g_c(t; y, v; x, u) = \Gamma_{\mathrm{OU}}(t; y, v; x, u) + \Gamma_{\mathrm{OU}}(t; y, v; -x, -u)\,.$$

Hypotheses 3.2
We consider the set of hypotheses ($H_{\textit{Reflected Even}}$):

($H_{\textit{Reflected Even}}$)-(i) $\mu_0\colon (x, u) \in (\mathcal{D} \times \mathbb{R}) \mapsto [0, 1]$ *is a probability measure with density that we also denote as* μ_0*, such that* $\partial_x\mu_0, \partial_{xx}\mu_0 \in L^1(\mathcal{D} \times \mathbb{R}) \cap L^\infty(\mathcal{D} \times \mathbb{R})$*, the integrals* $\int_{\mathcal{D}\times\mathbb{R}}(|u| + u^2)\mu_0(dx, du)$ *and* $\int_{\mathcal{D}\times\mathbb{R}}(|u| + u^2)|\partial_x\mu_0|(x, u)\, dxdu$ *are bounded. We also assume that* μ_0 *is zero on a neighbourhood of* $x = 0$ *and vanishes at infinity. The integrals* $\int_{\mathbb{R}} \sup_{y\in\mathcal{D}} |\mu_0(y, v)|\, dv$ *and* $\int_{\mathbb{R}} \sup_{y\in\mathcal{D}} |\partial_y\mu_0(y, v)|\, dv$ *are bounded.*

($H_{\textit{Reflected Even}}$)-(ii) $\mu\colon \mathcal{D} \mapsto \mathbb{R}$ *is bounded, continuous, the right-hand side derivative* $\mu'_+(0) = 0$ *and* $\mu', \mu'' \in L^\infty((0, +\infty))$.

Remark 3.3. *We extend* μ *in the negative domain as an even function, therefore ($H_{\textit{Reflected Even}}$)-(ii) is needed to obtain a continuous differentiable function.*

Lemma 3.4. *Assume ($H_{\textit{Reflected Even}}$) and let* ρ *be the solution of the mild equation (3.7). Then, for any* $T > 0$*:*

(i) $\sup_{t\in[0,T]} \|\partial_x\rho(t, x, u)\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma,T}$ *and* $\sup_{t\in[0,T]} \|\partial_{xx}\rho(t, x, u)\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma,T}\,,$

(ii) $\sup_{t\in[0,T]} \int_{\mathbb{R}} \rho(t, 0, u)\, du < C_{\mu_0,\mu,\sigma,T}\,,$

(iii) $\sup_{t\in[0,T]} \int_{\mathbb{R}} |\partial_x\rho(t, 0, u)|\, du < C_{\mu_0,\mu,\sigma,T}\,.$

Proof. The first item is proved by extending the solution of the reflected mild equation on the domain

$\mathcal{D}^c \times \mathbb{R}$. For any $(t,x,u) \in (0,+\infty) \times \mathcal{D} \times \mathbb{R}$:

$$
\begin{aligned}
2\rho(t,x,u) &= 2\int_{\mathcal{D}\times\mathbb{R}} g_c(t;y,v;x,u)\mu_0(y,v)\,dydv + 2\beta\int_0^t\int_{\mathcal{D}\times\mathbb{R}} \partial_v g_c(t-s;y,v;x,u)\mu(y)\rho(s,y,v)\,dsdydv \\
&= \int_{\mathcal{D}\times\mathbb{R}} g_c(t;y,v;x,u)\mu_0(y,v)\,dydv + \beta\int_0^t\int_{\mathcal{D}\times\mathbb{R}} \partial_v g_c(t-s;y,v;x,u)\mu(y)\rho(s,y,v)\,dsdydv \\
&\quad + \int_{(-\infty,0]\times\mathbb{R}} g_c(t;-z,-w;x,u)\mu_0(-z,-w)\,dzdw \\
&\quad + \beta\int_0^t\int_{(-\infty,0]\times\mathbb{R}} \partial_v g_c(t-s;-z,-w;x,u)\mu(-z)\rho(s,-z,-w)\,dsdzdw
\end{aligned}
\tag{3.8}
$$

where on the last two lines the change of variable $(y,v) \to (-z,-w)$ was performed. It is easy to see that for any $(t,y,v,x,u) \in \mathbb{R}^+ \times \mathbb{R}^4$,

$$
g_c(t;y,v;x,u) = g_c(t;y,v;-x,-u) = g_c(t;-y,-v;x,u) = g_c(t;-y,-v;-x,-u)\,.
$$

We extend

$$
\overline{\mu}_0(x,u) = \begin{cases} \mu_0(x,u) & \text{for } x \geq 0 \\ \mu_0(-x,-u) & \text{for } x < 0 \end{cases} \tag{3.9}
$$

while

$$
\overline{\rho}(t,x,u) = \begin{cases} \rho(t,x,u) & \text{for } x \geq 0 \\ \rho(t,-x,-u) & \text{for } x < 0\,. \end{cases} \tag{3.10}
$$

and

$$
\overline{\mu}(x) = \begin{cases} \mu(x) & \text{for } x \geq 0 \\ \mu(-x) & \text{for } x < 0 \end{cases} \tag{3.11}
$$

thus equation (3.8) becomes:

$$
2\rho(t,x,u) = \int_{\mathbb{R}^2} g_c(t;y,v;x,u)\overline{\mu}_0(y,v)\,dydv + \beta\int_0^t\int_{\mathbb{R}^2} \partial_v g_c(t-s;y,v;x,u)\overline{\mu}(y)\overline{\rho}(s,y,v)\,dsdydv\,. \tag{3.12}
$$

We can also notice then that we can use similar arguments as in Lemma 4.5 to show that $\sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} \rho(t,x,u)$ and $\sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} |\partial_x \rho(t,x,u)|$ are bounded (with bound that depends on β). This is because $\sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} \overline{\rho}(t,x,u) = \sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} \rho(t,x,u)$ and $\sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} |\partial_x \overline{\rho}(t,x,u)| = \sup_{(t,x,u)\in[0,T]\times\mathcal{D}\times\mathbb{R}} |\partial_x \rho(t,x,u)|$, so the same procedure as Lemma 4.5 using Gronwall's inequality can be used. These bounds are useful when applying various integration by parts against functions that vanish at infinity.

***(i)* Norm of the derivatives**

We prove that $\partial_x \rho(t,x,u) \in L^1(\mathcal{D}\times\mathbb{R})$.

$$\begin{aligned}
2\partial_x\rho(t,x,u) &= \int_{\mathbb{R}^2} \partial_x g_c(t;y,v;x,u)\overline{\mu}_0(y,v)\,dydv + \beta\int_0^t\int_{\mathbb{R}^2}\partial_v\partial_x g_c(t-s;y,v;x,u)\overline{\mu}(y)\overline{\rho}(s,y,v)\,dsdydv\\
&= \int_{\mathbb{R}^2}\left(\partial_x\Gamma_{\text{OU}}(t;y,v;x,u)+\partial_x\Gamma_{\text{OU}}(t;y,v;-x,-u)\right)\overline{\mu}_0(y,v)\,dydv\\
&\quad+\beta\int_0^t\int_{\mathbb{R}^2}\partial_v\left(\partial_x\Gamma_{\text{OU}}(t-s;y,v;x,u)+\partial_x\Gamma_{\text{OU}}(t-s;y,v;-x,-u)\right)\overline{\mu}(y)\overline{\rho}(s,y,v)\,dsdydv\\
&= -\int_{\mathbb{R}^2}\left(\Gamma_{\text{OU}}(t;y,v;x,u)-\Gamma_{\text{OU}}(t;y,v;-x,-u)\right)\partial_y\overline{\mu}_0(y,v)\,dydv\\
&\quad-\beta\int_0^t\int_{\mathbb{R}^2}\partial_v\left(\Gamma_{\text{OU}}(t-s;y,v;x,u)-\Gamma_{\text{OU}}(t-s;y,v;-x,-u)\right)\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\,dsdydv
\end{aligned}\tag{3.13}$$

thus ($H_{\textit{Reflected Even}}$), $\overline{\mu}$ and $\overline{\mu}_0$ are continuously differentiable:

$$\begin{aligned}
&2\left\|\partial_x\rho(t,\cdot,\cdot)\right\|_{L^1(\mathcal{D}\times\mathbb{R})}\leq 2\left\|\partial_y\overline{\mu}_0\right\|_{L^1(\mathbb{R}^2)}\\
&\quad+\beta\int_0^t\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|\partial_v\Gamma_{\text{OU}}(t-s;y,v;x,u)\right|+\left|\partial_v\Gamma_{\text{OU}}(t-s;y,v;-x,-u)\right|\right)dxdu\,\left|\partial_y(\overline{\mu}\overline{\rho}(s,y,v)\right|\,dsdydv\,.
\end{aligned}\tag{3.14}$$

By the bound (2.66) and Corollary 4.2, we obtain that:

$$\begin{aligned}
&2\left\|\partial_x\rho(t,\cdot,\cdot)\right\|_{L^1(\mathcal{D}\times\mathbb{R})}\leq 2\left\|\partial_y\overline{\mu}_0\right\|_{L^1(\mathbb{R}^2)}+4\beta\int_0^t C(\beta,\sigma,t-s)\left\|\partial_y\left(\mu(y)\rho(s,\cdot,\cdot)\right)\right\|_{L^1(\mathcal{D}\times\mathbb{R})}\\
&\leq 2\left\|\partial_y\overline{\mu}_0\right\|_{L^1(\mathbb{R}^2)}+4C_\sigma\int_0^t\frac{1}{\sqrt{t-s}}\left\|\partial_y\left(\mu(y)\rho(s,\cdot,\cdot)\right)\right\|_{L^1(\mathcal{D}\times\mathbb{R})}
\end{aligned}\tag{3.15}$$

and by Gronwall's inequality and Lemma 2.7, we obtain that

$$\sup_{t\in[0,T]}\left\|\partial_x\rho(t,x,u)\right\|_{L^1(\mathcal{D}\times\mathbb{R})}<C_{\mu_0,\mu,\sigma,T}$$

uniformly in β.

For β fixed, we extend, by continuity, $\partial_x\rho(t,x,u)$ up to the boundary $x=0$. Since $\partial_x\overline{\rho}(t,x,u)=\partial_x\overline{\rho}(t,-x,-u)$, we have that $\partial_x\overline{\rho}(t,x,u)$ is continuous on $\mathbb{R}\times\mathbb{R}$.

So, by similar arguments to Lemma 2.7, since $\partial_{xx}\overline{\mu}_0\in L^\infty(\mathbb{R}^2)$, then $\partial_{xx}\rho(t,x,u)$ is bounded in $L^1(\mathcal{D}\times\mathbb{R})$, uniformly in β.

***(ii)* Norm of the trace of the density**

For any $(t,u) \in (0,+\infty)\times\mathbb{R}$, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}} \rho(t,0,u)\,du = \int_{\mathcal{D}\times\mathbb{R}} \int_{\mathbb{R}} g_c(t;y,v;0,u)\,du \mu_0(y,v)\,dydv \\
&\quad + \beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \partial_v \int_{\mathbb{R}} g_c(t-s;y,v;0,u)\,du \mu(y)\rho(s,y,v)\,dsdydv \\
&= \int_{\mathcal{D}\times\mathbb{R}} 2M(t;y,v;0)\mu_0(y,v)\,dydv + 2\beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \partial_v M(t-s;y,v;0)\mu(y)\rho(s,y,v)\,dsdydv \\
&= \int_{\mathcal{D}\times\mathbb{R}} 2M(t;y,v;0)\mu_0(y,v)\,dydv + 2\int_0^t (1-e^{-\beta(t-s)}) \int_{\mathcal{D}\times\mathbb{R}} \partial_y M(t-s;y,v;0)\mu(y)\rho(s,y,v)\,dsdydv \\
&= \int_{\mathcal{D}\times\mathbb{R}} 2M(t;y,v;0)\mu_0(y,v)\,dydv + 2\mu(0)\int_0^t (1-e^{-\beta(t-s)}) \int_{\mathbb{R}} M(t-s;0,v;0)\rho(s,0,v)\,dsdv \\
&\quad - 2\int_0^t (1-e^{-\beta(t-s)}) \int_{\mathcal{D}\times\mathbb{R}} M(t-s;y,v;0)\partial_v\left(\mu(y)\rho(s,y,v)\right)\,dsdydv
\end{aligned}
\tag{3.16}
$$

where M the marginal density of the position defined in (2.32), obtained from the joint transitional density Γ_{OU}. By ($H_{\textit{Reflected Even}}$), we have that:

$$
\begin{aligned}
\left|\int_{\mathbb{R}} \rho(t,0,u)\,du\right| \le 2\int_{\mathbb{R}} \sup_{y\in\mathcal{D}} \mu_0(y,v)\,dv + \sqrt{\frac{2}{\pi}}\left(\|\partial_y\mu\|_{L^\infty(\mathbb{R})} + \|\mu\|_{L^\infty(\mathbb{R})}\|\partial_y\rho\|_{L^1(\mathcal{D}\times\mathbb{R})}\right)\int_0^t \frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}(t-s)}\,ds \\
+ \sqrt{\frac{2}{\pi}}\int_0^t \frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}(t-s)}\left(\int_{\mathbb{R}} \rho(s,0,v)\,dv\right)ds
\end{aligned}
\tag{3.17}
$$

By the bound (A.19) we have that for any $T \ge t$:

$$
\begin{aligned}
\left|\int_{\mathbb{R}} \rho(t,0,u)\,du\right| \le 2\int_{\mathbb{R}} \sup_{y\in\mathcal{D}} \mu_0(y,v)\,dv + \sqrt{\frac{2}{\pi}}\frac{C\sqrt{T}}{\sigma}\left(\|\partial_y\mu\|_{L^\infty(\mathbb{R})} + \|\mu\|_{L^\infty(\mathbb{R})}\|\partial_y\rho\|_{L^1(\mathcal{D}\times\mathbb{R})}\right) \\
+ \frac{C_\sigma\sqrt{2}}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\left(\int_{\mathbb{R}} \rho(s,0,v)\,dv\right)ds
\end{aligned}
$$

and Gronwall's Lemma as presented in the Remark 4.4, then we can bound uniformly in β, for any $T > 0$

$$
\sup_{t\in[0,T]} \int_{\mathbb{R}} \rho(t,0,u)\,du \le C_{\mu_0,\mu,\sigma,T}\,. \tag{3.18}
$$

(iii) **Norm of the trace of the derivative of the density**

We go back to equation (3.13):

$$
\begin{aligned}
&2\partial_x\rho(t,x,u) = -\int_{\mathbb{R}^2} \left(\Gamma_{\text{OU}}(t;y,v;x,u) - \Gamma_{\text{OU}}(t;y,v;-x,-u)\right)\partial_y\overline{\mu}_0(y,v)\,dydv \\
&\quad - \beta\int_0^t \int_{\mathbb{R}^2} \partial_v\left(\Gamma_{\text{OU}}(t-s;y,v;x,u) - \Gamma_{\text{OU}}(t-s;y,v;-x,-u)\right)\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\,dsdydv \\
&= -\int_{\mathbb{R}^2} \left(\Gamma_{\text{OU}}(t;y,v;x,u) - \Gamma_{\text{OU}}(t;y,v;-x,-u)\right)\partial_y\overline{\mu}_0(y,v)\,dydv \\
&\quad + \int_0^t (1-e^{-\beta(t-s)}) \int_{\mathbb{R}^2} \left(\Gamma_{\text{OU}}(t-s;y,v;x,u) - \Gamma_{\text{OU}}(t-s;y,v;-x,-u)\right)\partial_{yy}\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\,dsdydv
\end{aligned}
\tag{3.19}
$$

thus

$$2\int_{\mathbb{R}} |\partial_x\rho(t,0,u)|\, du \le 2\int_{\mathbb{R}} \sup_{y\in\mathbb{R}} |\partial_y\overline{\mu}_0(y,v)|\, dv + 2\sup_{s\in[0,t]} \|\partial_{yy}\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\|_{L^1(\mathbb{R}^2)} \int_0^t \frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}(t-s)}\, ds\,. \tag{3.20}$$

We can conclude by (A.19) that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} |\partial_x\rho(t,0,u)|\, du \le C_{\mu_0,\mu,\sigma,T} \tag{3.21}$$

uniformly in β, for any $T>0$.

■

Lemma 3.5. *Assume ($H_{\text{Reflected Even}}$) and consider ρ the solution to the mild equation (3.7), then we have the following controls for any $T \ge t$:*

(i) $\|u\rho(t,x,u)\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma}\sqrt{\beta}$ *and* $\left\|u^2\rho(t,x,u)\right\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma}\beta\,,$

(ii) $\sup_{t\in[0,T]} \left\|u^2\partial_x\rho(t,x,u)\right\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma,T}\beta\,,$

(iii) $\sup_{t\in[0,T]} \left\|\int_{\mathbb{R}} u\rho(t,x,u)\, du\right\|_{L^1(\mathcal{D})} \le C_{\mu_0,\mu,\sigma,T}\,.$

Proof. ***(i)*** **Norm of first and second moment**

Since the integrated term is positive and by (A.4) $\Sigma_{uu}(t) \le \sigma\sqrt{\beta}$, we have that:

$$\begin{aligned}
&\int_{\mathcal{D}} \left|\int_{\mathbb{R}} |u|\rho(t,x,u)\, du\right| dx = \int_{\mathcal{D}}\int_{\mathbb{R}} |u|\rho(t,x,u)\, du\, dx = \int_{\mathbb{R}} |u| \int_{\mathcal{D}} \rho(t,x,u)\, dxdu \\
&= \int_{\mathcal{D}\times\mathbb{R}} \left(\int_{\mathcal{D}\times\mathbb{R}} |u| g_c(t;y,v;x,u)\, dxdu\right) \mu_0(dy,dv) \\
&\quad + \beta\int_0^t \int_{\mathcal{D}\times\mathbb{R}} \partial_v \left(\int_{\mathcal{D}\times\mathbb{R}} |u| g_c(t-s;y,v;x,u)\, dxdu\right) \rho(s,y,v)\mu(y)\, dydvds\,.
\end{aligned} \tag{3.22}$$

By performing the change of variable $(x,u)\to(-x,-u)$ then

$$\begin{aligned}
&\int_{\mathcal{D}\times\mathbb{R}} |u| g_c(t;y,v;x,u)\, dxdu = \int_{\mathcal{D}\times\mathbb{R}} |u| \left(\Gamma_{\text{OU}}(t;y,v;x,u) + \Gamma_{\text{OU}}(t;y,v;-x,-u)\right)\, dxdu \\
&= \int_{\mathbb{R}^2} |u|\Gamma_{\text{OU}}(t;y,v;x,u)\, dxdu\,.
\end{aligned} \tag{3.23}$$

So by similar calculations to those in Lemma 2.8

$$\|u\rho(t,x,u)\|_{L^1(\mathcal{D}\times\mathbb{R})} \le C_{\mu_0,\mu,\sigma}\sqrt{\beta}\,, \tag{3.24}$$

and also

$$\left\|u^2\rho(t,x,u)\right\|_{L^1(\mathcal{D}\times\mathbb{R})} \le C_{\mu_0,\mu,\sigma}\beta\,. \tag{3.25}$$

(ii) **Second moment and derivative**

We recall that we have

$$2\partial_x\rho(t,x,u) = -\int_{\mathbb{R}^2} \left(\Gamma_{\rm OU}(t;y,v;x,u) - \Gamma_{\rm OU}(t;y,v;-x,-u)\right)\partial_y\overline{\mu}_0(y,v)\,dydv$$
$$-\beta\int_0^t\int_{\mathbb{R}^2}\partial_v\left(\Gamma_{\rm OU}(t-s;y,v;x,u) - \Gamma_{\rm OU}(t-s;y,v;-x,-u)\right)\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\,dsdydv \tag{3.26}$$

thus, we have

$$\begin{aligned}
2\int_{\mathcal{D}\times\mathbb{R}} u^2\left|\partial_x\rho(t,x,u)\right|\,dxdu &\le \int_{\mathcal{D}\times\mathbb{R}} u^2\int_{\mathbb{R}^2}\Gamma_{\rm OU}(t;y,v;x,u)\left|\partial_y\overline{\mu}_0(y,v)\right|\,dydvdxdu\\
&+\int_{\mathcal{D}\times\mathbb{R}} u^2\int_{\mathbb{R}^2}\Gamma_{\rm OU}(t;y,v;-x,-u)\left|\partial_y\overline{\mu}_0(y,v)\right|\,dydvdxdu\\
&+\beta\int_0^t\int_{\mathcal{D}\times\mathbb{R}} u^2\int_{\mathbb{R}^2}\left|\partial_v\Gamma_{\rm OU}(t-s;y,v;x,u)\right|\left|\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\right|\,dsdydvdxdu\\
&+\beta\int_0^t\int_{\mathcal{D}\times\mathbb{R}} u^2\int_{\mathbb{R}^2}\left|\partial_v\Gamma_{\rm OU}(t-s;y,v;-x,-u)\right|\left|\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\right|\,dsdydvdxdu\,.
\end{aligned} \tag{3.27}$$

Since the functions we are integrating are positive, the bound remains valid when extending the integrals from $x\in\mathcal{D}$ to $x\in\mathbb{R}$. For the second and fourth term of the r.h.s., we perform the change of variable $x\mapsto -x$ and $u\mapsto -u$, thus obtaining

$$\begin{aligned}
2\int_{\mathcal{D}\times\mathbb{R}} u^2\left|\partial_x\rho(t,x,u)\right|\,dxdu &\le 2\int_{\mathbb{R}^4} u^2\Gamma_{\rm OU}(t;y,v;x,u)\left|\partial_y\overline{\mu}_0(y,v)\right|\,dydvdxdu\\
&+2\beta\int_0^t\int_{\mathbb{R}^4} u^2\left|\partial_v\Gamma_{\rm OU}(t-s;y,v;x,u)\right|\left|\partial_y\left(\overline{\mu}(y)\overline{\rho}(s,y,v)\right)\right|\,dsdydvdxdu
\end{aligned} \tag{3.28}$$

and we can notice that we can apply the same arguments as in the proof of Lemma 2.8-*(iii)* to obtain that for any $T>0$

$$\sup_{t\in[0,T]}\left\|u^2\partial_x\rho(t,x,u)\right\|_{L^1(\mathcal{D}\times\mathbb{R})} < C_{\mu_0,\mu,\sigma,T}\,\beta$$

since by the definition of $\overline{\rho}$ in (3.10), we have that $\int_{\mathbb{R}^2}\overline{\rho}(t,x,u)\,dxdu = 2\int_{\mathcal{D}\times\mathbb{R}}\rho(t,x,u)\,dxdu$ and $\int_{\mathbb{R}^2}\left|\partial_x\overline{\rho}(t,x,u)\right|\,dxdu = 2\int_{\mathcal{D}\times\mathbb{R}}\left|\partial_x\rho(t,x,u)\right|\,dxdu$.

***(iii)* Norm of the first moment**

We calculate the bound

$$\begin{aligned}
\int_{\mathbb{R}} u\rho(t,x,u)\,du &= \int_{\mathcal{D}\times\mathbb{R}}\left(\int_{\mathbb{R}} ug_c(t;y,v;x,u)\,du\right)\mu_0(dy,dv)\\
&\quad+\beta\int_0^t\int_{\mathcal{D}\times\mathbb{R}}\partial_v\left(\int_{\mathbb{R}} ug_c(t-s;y,v;x,u)\,du\right)\rho(s,y,v)\mu(y)\,dydvds\,.
\end{aligned} \tag{3.29}$$

By previous calculations, we have that

$$\begin{aligned}
\int_{\mathbb{R}} ug_c(t;y,v;x,u)\,du &= \rho(t)\Sigma_{uu}(t)\Sigma_{xx}(t)\partial_y\left(M(t;y,v;x) - M(t;y,v;-x)\right)\\
&\quad+ve^{-\beta t}\left(M(t;y,v;x) - M(t;y,v;-x)\right)
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
\partial_v\int_{\mathbb{R}} ug_c(t;y,v;x,u)\,du &= \rho(t)\Sigma_{uu}(t)\Sigma_{xx}(t)\frac{1-e^{-\beta t}}{\beta}\partial_{yy}\left(M(t;y,v;x) - M(t;y,v;-x)\right)\\
&+e^{-\beta t}\left(M(t;y,v;x) - M(t;y,v;-x)\right) + \frac{v}{\beta}(1-e^{-\beta t})e^{-\beta t}\partial_y\left(M(t;y,v;x) - M(t;y,v;-x)\right)
\end{aligned} \tag{3.31}$$

The terms that correspond to the second and third of this equality are treated similarly as in the proof of Lemma 2.7. For the term that corresponds to the first one of this equality, we transfer on the density ρ, the two partial derivatives in y through integration by parts. These i.b.p. produce boundary terms, which are bounded in $L^1(\mathcal{D})-$norm as following:

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}} \rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)})\partial_y \left(M(t-s;0,v;x)-M(t-s;0,v;-x)\right)\rho(s,0,v)\mu(0)dvds \right\|_{L^1(\mathcal{D})} \\
&\leq 2|\mu(0)| \int_0^t \rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)}) \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\partial_y M(t-s;0,v;x)| \right) \rho(s,0,v)\, dvds \\
&\leq 2|\mu(0)|\sigma^2 C_{\mu_0,\mu,\sigma,t} \int_0^t \frac{1-e^{-\beta(t-s)}}{\Sigma_{xx}(t-s)}\, ds \leq 2|\mu(0)|\sigma^2 C_{\mu_0,\mu,\sigma,T} C_{\sigma,T}
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}} \rho(t-s)\Sigma_{uu}(t-s)\Sigma_{xx}(t-s)(1-e^{-\beta(t-s)}) \left(M(t-s;0,v;x)-M(t-s;0,v;-x)\right) \partial_y \left(\rho(s,0,v)\mu(0)\right) dvds \right\|_{L^1(\mathcal{D})} \\
&\leq 2\sigma^2|\mu(0)|\, \|\partial_y \rho(t,0,v)\|_{L^1(\mathbb{R})} \leq 2\sigma^2|\mu(0)| C_{\mu_0,\mu,\sigma,T}\,,
\end{aligned} \tag{3.33}$$

where we have used the bound (A.19), the fact that $\mu'_+(0) = 0$ and Lemma 3.4 to control the norms of the trace of the density and its partial derivative. We can see that the boundary terms can be controlled uniformly in β. For all the other terms, we obtain similar bounds uniform in β by following the same arguments as in Lemma (2.7).

■

The mild equation for the comparison process

Let $(Y_t^f)_{t\geq 0}$ defined as the solution of the SDE for $x_0 > 0$:

$$\left\{ Y_t^f = x_0 + \int_0^t \operatorname{sign}(Y_s^f)\mu(|Y_s^f|)\, ds + \sigma W_t \right. \tag{3.34}$$

and define $(Y_t)_{t\geq 0} = (|Y_t^f|)_{t\geq 0}$. We have by Tanaka's formula that

$$Y_t = x_0 + \int_0^t \mu(|Y_s^f|)\, ds + \sigma \int_0^t \operatorname{sign}(Y_s^f)\, dW_s + L_t^{Y^f} \tag{3.35}$$

where $(L_t^{Y^f})_{t\geq 0}$ is the local time at zero of the process $(Y_t^f)_{t\geq 0}$. We also introduce the process $(\widetilde{Y}_t^f)_{t\geq 0}$ which solves the SDE 3.34 for $\mu \equiv 0$ and similarly $(\widetilde{Y}_t)_{t\geq 0} = (|\widetilde{Y}_t^f|)_{t\geq 0}$. The transition density of $(\widetilde{Y}_t)_{t\geq 0}$ is

$$\widetilde{g}\colon (t,y,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \frac{1}{\sqrt{2\pi\sigma^2 t}} \left(\exp\left(-\frac{(x-y)^2}{2\sigma^2 t}\right) + \exp\left(-\frac{(x+y)^2}{2\sigma^2 t}\right) \right)$$

or by using the function Γ_{B} defined in the previous section then $\widetilde{g}\colon (t,y,x) \mapsto \Gamma_{\mathrm{B}}(t;y;x) + \Gamma_{\mathrm{B}}(t;y;-x)$. We introduce the operator $\overline{S}_t(f)(y) = \mathbb{E}_y f(\widetilde{Y}_t) = \int_{\mathcal{D}} \widetilde{g}(t,y,x) f(x)\, dx$ and its adjoint $\overline{S}_t^*(\mu_0)(x) = \int_{\mathcal{D}} \widetilde{g}(t,y,x)\mu_0(y)\, dy$. Consider the function $G_{t,f}\colon (s,y) \in [0,t) \times \mathcal{D} \mapsto \overline{S}_{t-s}(f)(y)$, which is a solution to the PDE

$$
\begin{cases}
\partial_s G_{t,f} + \frac{\sigma^2}{2}\partial_{xx} G_{t,f} = 0, & \text{on } [0,t) \times \mathcal{D}\,, \\
\lim\limits_{y \longrightarrow 0} \partial_y G_{t,f}(s,y) = 0, & \text{on } [0,t) \\
\lim\limits_{s \to t^-} G_{t,f}(s,y) = f(y), & \text{on } \mathcal{D}\,.
\end{cases}
\tag{3.36}
$$

then we have that:

$$
\begin{aligned}
\mathbb{E}_{\mu_0^Y} G_{t,f}(t,Y_t) &= \mathbb{E}_{\mu_0^Y} G_{t,f}(0,Y_0) + \mathbb{E}_{\mu_0^Y}\int_0^t \partial_s G_{t,f}(s,Y_s)\,ds + \mathbb{E}_{\mu_0^Y}\int_0^t \mu(Y_s)\partial_y G_{t,f}(s,Y_s)\,ds \\
&\quad + \mathbb{E}_{\mu_0^Y}\int_0^t \operatorname{sign}(Y_s^f)\partial_y G_{t,f}(s,Y_s)\,dW_s + \mathbb{E}_{\mu_0^Y}\int_0^t \partial_y G_{t,f}(s,Y_s)\,dL_s^{Y^f} \\
&\quad + \frac{\sigma^2}{2}\mathbb{E}_{\mu_0^Y}\int_0^t \partial_{yy} G_{t,f}(s,Y_s)\,ds \\
&= \mathbb{E}_{\mu_0^Y} G_{t,f}(0,Y_0) + \mathbb{E}_{\mu_0^Y}\int_0^t \mu(Y_s)\partial_y G_{t,f}(s,Y_s)\,ds\,.
\end{aligned}
\tag{3.37}
$$

We recall that $L_t^{Y^f}$ is the local time of Y_t^f at 0, thus it only increases on the set $\{Y_t^f = 0\}$ and it is constant anywhere else. By the definition of the process $(Y_t)_{t\geq 0}$, we have that $\{Y_t^f = 0\} = \{Y_t = 0\}$ but by the boundary condition in the PDE (3.36), so $\partial_y G_{t,f}(s,Y_s) = 0$ on $\{Y_t = 0\}$. This implies that the integral w.r.t. the local time in (3.37) is zero.

Further developing (3.37) allows us to obtain the following mild equation for $p\colon \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^+$ the density of the process $(Y_t)_{t\geq 0}$ with initial condition $\mu_0^Y = \int_{\mathbb{R}} \mu_0(y,v)\,dv$:

$$
p(t,x) = \overline{S}_t^*(\mu_0^Y)(x) + \int_0^t \int_{\mathcal{D}} \partial_y \widetilde{g}(t-s,y,x)\mu(y)p(s,y)\,dyds\,.
\tag{3.38}
$$

3.3 Bounding the difference

Lemma 3.6. *Assume (H_{Forward}), ($H_{\text{Reflected Forward}}$)-$(iii)$ and ($H_{\text{Reflected Forward}}$)-$(iv)$ are verified and assume ρ and p are solutions to the mild equations (3.7) and, respectively, (3.38). Then, for large enough β, we have that*

$$
\left\| \int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x) \right\|_{L^1(\mathcal{D})} \leq C_{\mu_0,\mu,\sigma,t}\frac{\ln(\beta)}{\beta}\,.
\tag{3.39}
$$

Let us consider the difference between the solution of the mild equation that corresponds to the reflected Langevin process (3.7) and the solution of the mild equation for the comparison process (3.38)

$$
\begin{aligned}
&\int_{\mathbb{R}} \rho(t,x,u)\,du - p(t,x) = \int_{\mathcal{D}\times\mathbb{R}^2} g_c(t;y,v;x,u)\mu_0(y,v)\,dydvdu - \overline{S}^*_t(\mu_0^Y)(x) \\
&\quad + \beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}^2} \partial_v g_c(t-s;y,v;x,u)\mu(y)\rho(s,y,v)\,dsdydvdu - \int_0^t \int_{\mathcal{D}} \partial_y \widetilde{g}(t-s,y,x)\mu(y)p(s,y)\,dyds\,. \\
&= \int_{\mathcal{D}\times\mathbb{R}^2} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\mu_0(y,v)\,dydvdu - \int_{\mathcal{D}} \Gamma_{\mathrm{B}}(t;y;x)\mu_0^Y(y)\,dy \\
&\quad + \int_{\mathcal{D}\times\mathbb{R}^2} \Gamma_{\mathrm{OU}}(t;y,v;-x,-u)\mu_0(y,v)\,dydvdu - \int_{\mathcal{D}} \Gamma_{\mathrm{B}}(t;y;-x)\mu_0^Y(y)\,dy \\
&\quad + \beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}^2} \partial_v \Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\mu(y)\rho(s,y,v)\,dsdydvdu - \int_0^t \int_{\mathcal{D}} \partial_y \Gamma_{\mathrm{B}}(t-s;y;x)\mu(y)p(s,y)\,dyds \\
&\quad + \beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}^2} \partial_v \Gamma_{\mathrm{OU}}(t-s;y,v;-x,-u)\mu(y)\rho(s,y,v)\,dsdydvdu - \int_0^t \int_{\mathcal{D}} \partial_y \Gamma_{\mathrm{B}}(t-s;y;-x)\mu(y)p(s,y)\,dyds\,.
\end{aligned}
\tag{3.40}
$$

The first two terms correspond to the initial condition. Because for any $v \in \mathbb{R}$, $\mu_0(0,v) = 0$, simplifying any i.b.p., the same arguments presented in the section Bounding the initial terms for the proof of Theorem 2.5 apply to obtain a similar control

$$
\left\| \int_{\mathcal{D}\times\mathbb{R}^2} g_c(t;y,v;x,u)\mu_0(y,v)\,dydvdu - \overline{S}^*_t(\mu_0^Y)(x) \right\|_{L^1(\mathcal{D})} \leq C_{\mu_0,\mu,\sigma,t}\frac{1}{\beta}\,. \tag{3.41}
$$

The section Bounding time integral of the proof of Theorem 2.5 contains the main arguments needed to obtain the needed bounds for the last two differences in (3.40), some analysis needs to be carried out for the boundary terms though, when performing integration by parts. We can notice that the two last differences in (3.40) are very similar and the change of sign does not affect the arguments, therefore we present the steps just for one of the terms.

Similar to formula (2.40) we have

$$
\begin{aligned}
&\beta \int_0^t \int_{\mathcal{D}\times\mathbb{R}^2} \partial_v \Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\mu(y)\rho(s,y,v)\,dsdydvdu - \int_0^t \int_{\mathcal{D}} \partial_y \Gamma_{\mathrm{B}}(t-s;y;x)\mu(y)p(s,y)\,dyds \\
&= \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \mu(y) \left((1-e^{-\beta(t-s)})\partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x) - \partial_y g(y, \sigma^2(t-s), x) \right) \rho(s,y,v)\,dydvds \\
&\quad + \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \mu(y)(1-e^{-\beta(t-s)})\rho(s,y,v)\,dydvds \times \\
&\qquad \times \left(\partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \Sigma^2_{xx}(t-s), x) - \partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x) \right).
\end{aligned}
\tag{3.42}
$$

and, thus, we have once more the sum of two terms, one that corresponds to the difference between two Gaussians with different means and the other that corresponds to the difference between two Gaussians with different variances.

Difference between two ex-centred Gaussians in (3.42) By applying Taylor's expansion with integral remainder, we obtain that

$$
\begin{aligned}
&\int_0^t \int_{\mathcal{D}\times\mathbb{R}} \mu(y) \left((1-e^{-\beta(t-s)})\partial_y g\left(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x\right) - \partial_y g(y, \sigma^2(t-s), x) \right) \rho(s,y,v)\, dydvds \\
&= -\int_0^t e^{-\beta(t-s)} \int_{\mathcal{D}} \mu(y)\partial_y g(y, \sigma^2(t-s), x) \left(\int_{\mathbb{R}} \rho(s,y,v)\, dv \right) dyds \\
&\quad + \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \frac{v}{\beta}\mu(y)(1-e^{-\beta(t-s)})^2 \partial_{yy} g\left(y, \sigma^2(t-s), x\right) \rho(s,y,v)\, dvdyds \\
&\quad + \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \frac{v^2}{\beta^2}\mu(y)(1-e^{-\beta(t-s)})^3 \int_0^1 (1-\theta)\partial_{yyy} g\left(y + \theta\frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), x\right) d\theta \rho(s,y,v)\, dydvds\,.
\end{aligned}
\tag{3.43}
$$

The first term of the r.h.s. of the equality (3.43) is transformed by an i.b.p. and taking the L^1-norm to obtain

$$
\begin{aligned}
&\left\| \int_0^t e^{-\beta(t-s)} \int_{\mathcal{D}} \mu(y)\partial_y g(y, \sigma^2(t-s), x) \left(\int_{\mathbb{R}} \rho(s,y,v)\, dv \right) dyds \right\|_{L^1(\mathcal{D})} \\
&\le |\mu(0)| \int_0^t e^{-\beta(t-s)} \left(\int_{\mathcal{D}} g(0, \sigma^2(t-s), x)\, dx \right) \left(\int_{\mathbb{R}} \rho(s,0,v)\, dv \right) dyds \\
&\quad + \left\| \int_0^t e^{-\beta(t-s)} \int_{\mathcal{D}} g(y, \sigma^2(t-s), x) \left(\int_{\mathbb{R}} \partial_y \left(\mu(y)\rho(s,y,v)\right)\, dv \right) dyds \right\|_{L^1(\mathcal{D})} \\
&\le |\mu(0)| C_{\mu_0,\mu,\sigma,t} \int_0^t e^{-\beta(t-s)}\, ds + \|\partial_y \left(\mu(y)\rho(\cdot,y,v)\right)\|_{L^1(\mathcal{D})} \int_0^t e^{-\beta(t-s)}\, ds \le C_{\mu_0,\mu,\sigma,t}\frac{1}{\beta}\,,
\end{aligned}
\tag{3.44}
$$

where the bounds of Lemma 3.4 have been used.

For the second term of the r.h.s. of the equality (3.43), we use the same arguments as in Theorem 2.5 and the bounds in Lemma 3.5, to obtain that

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathcal{D}\times\mathbb{R}} \frac{v}{\beta}\mu(y)(1-e^{-\beta(t-s)})^2 \partial_{yy} g\left(y, \sigma^2(t-s), x\right) \rho(s,y,v)\, dvdyds \right\|_{L^1(\mathcal{D})} \\
&\le \frac{1}{\beta} \|\mu\|_{L^\infty} C_{\mu_0,\mu,\sigma,t} \int_0^t \frac{(1-e^{-\beta s})^2}{\sigma^2 s}\, ds \le C_{\mu_0,\mu,\sigma,t}\frac{\ln(\beta)}{\beta}\,.
\end{aligned}
\tag{3.45}
$$

Difference between two Gaussians with different variances in (3.42) To bound these terms, we use them same techniques of extending the equation on the whole domain $\mathbb{R}\times\mathbb{R}$ as in the proof of Lemma 3.4. Since we have sufficient regularity on this extension, we apply the same techniques as in Theorem 2.5 to obtain the same bound.

From the last difference of the r.h.s. of (3.40), we obtain the counterpart of the variance terms in (3.42), which we write below as:

$$
\begin{aligned}
&\int_0^t \int_{\mathcal{D}\times\mathbb{R}} \mu(y)(1-e^{-\beta(t-s)})\rho(s,y,v)\, dydvds \Big(\partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \Sigma_{xx}^2(t-s), -x) \\
&\qquad - \partial_y g(y + \frac{v}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), -x) \Big) \\
&= \int_0^t \int_{(-\infty,0]\times\mathbb{R}} \mu(-z)(1-e^{-\beta(t-s)})\rho(s,-z,-w)\, dzdwds \Big(\partial_y g(-z + \frac{-w}{\beta}(1-e^{-\beta(t-s)}), \Sigma_{xx}^2(t-s), -x) \\
&\qquad - \partial_y g(-z + \frac{-w}{\beta}(1-e^{-\beta(t-s)}), \sigma^2(t-s), -x) \Big)
\end{aligned}
\tag{3.46}
$$

by performing the change of variable $(y,v)\to(-z,-w)$. Since $g(y,\cdot,x)=g(-y,\cdot,-x)$, we sum the obtained result (3.46) to the variance terms in (3.42) and utilise the extensions for ρ (3.10) and for μ (3.11) to obtain that:

$$\begin{aligned}
&\int_0^t\int_{\mathcal{D}\times\mathbb{R}}\mu(y)(1-e^{-\beta(t-s)})\rho(s,y,v)\,dydvds\Big(\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\Sigma_{xx}^2(t-s),x)\\
&\qquad-\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x)\Big)\\
&+\int_0^t\int_{\mathcal{D}\times\mathbb{R}}\mu(y)(1-e^{-\beta(t-s)})\rho(s,y,v)\,dzdwds\Big(\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\Sigma_{xx}^2(t-s),-x)\\
&\qquad-\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),-x)\Big)\\
&=\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\overline{\mu}(y)(1-e^{-\beta(t-s)})\overline{\rho}(s,y,v)\,dydvds\Big(\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\Sigma_{xx}^2(t-s),x)\\
&\qquad-\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x)\Big)
\end{aligned}\tag{3.47}$$

By the controls on the various moments of the density proven in Lemma 3.4, we can apply the same techniques as in the Theorem 2.5 to obtain that:

$$\begin{aligned}
&\Big\|\int_0^t\int_{\mathbb{R}\times\mathbb{R}}\overline{\mu}(y)(1-e^{-\beta(t-s)})\overline{\rho}(s,y,v)\,dydvds\Big(\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\Sigma_{xx}^2(t-s),x)\\
&\qquad-\partial_y g(y+\frac{v}{\beta}(1-e^{-\beta(t-s)}),\sigma^2(t-s),x)\Big)\Big\|_{L^2(\mathbb{R})}\le C_{\mu,\mu_0,\sigma,t}\frac{1}{\beta}\,.
\end{aligned}\tag{3.48}$$

Final steps

We can notice that we have obtained the same controls on the different components of the difference of mild equations (3.40) as in Theorem 2.5, therefore we obtain the same bound and conclude the sketch of the proof for Lemma 3.6.

4 Conclusion and perspectives

We have seen that under ($H_{\textit{Weak Bound}}$), the weak error between the position process of a Langevin process and a corresponding uniformly elliptic diffusion in the Smoluchowski-Kramers limit decreases at least as $\dfrac{1}{\beta^{1-\varepsilon}}$ for any $\varepsilon>0$.

Similarly, we obtain the same bound on the error by introducing a specular reflection border for the Langevin process and instantaneous reflection on the uniform elliptic diffusion, provided we respect either condition ($H_{\textit{Reflected Odd}}$) or condition ($H_{\textit{Reflected Even}}$).

The most obvious extension would be to obtain this result for any type of drift, but we recall that the specular boundary condition is not linear. Another interesting result would be to obtain that a linear decrease of the error and also a Richardson-Romberg extrapolation for the error, extend the results to higher dimensions and different boundary domains.

Appendices

Appendix A

Simulation

1 Test case

We perform several numerical experiments in order to gauge if the theoretical bound on the weak error between the position reflected Langevin process (1.6) and reflected Brownian with drift (1.7).

We discretise the reflected Langevin process using the scheme already presented in the previous chapters. For the reflected Brownian with drift, we consider a symmetrised Euler scheme such as in [Bossy *et al.*, 2004] (we simulate the process at discretisation times $(t_i)_{i\geq 0}$ using a regular Euler scheme, and if the process escapes the domain $(0, +\infty)$, it is reintroduced in the domain by a symmetrisation around the origin). We denote by $(\bar{x}_t^{\beta,\Delta t}, \bar{u}_t^{\beta,\Delta t})_{t\geq 0}$ and $(\bar{Y}_t^{\Delta t})_{t\geq 0}$ the time-discretisation of these processes.

For the test function, we consider $f\colon x \mapsto x^2$. Concerning the drift function, μ we consider the functions:

- Case Odd: $\mu\colon x \in \mathbb{R}^+ \mapsto x\,,$
- Case Even: $\mu\colon x \in \mathbb{R}^+ \mapsto 1\,,$
- Case General: $\mu\colon x \in \mathbb{R}^+ \mapsto 1 + x\,.$

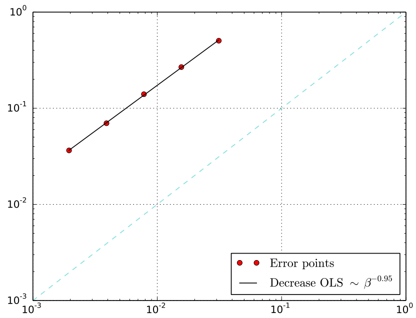
It can be see that the drift in Case General does not verify ($H_{\textit{Reflected Even}}$)-$(ii)$ or ($H_{\textit{Reflected Odd}}$)-$(ii)$. We set the discretisation time-step to $\Delta t = 2^{-11}$, $T = 1$ and we plot the function

$$\overline{\text{ErrorSK}}[f]\colon \beta \mapsto \left| \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} \Big(f(\bar{x}_T^{\beta,\Delta t,i}) - f(\bar{Y}_T^{\Delta t,i}) \Big) \right|$$

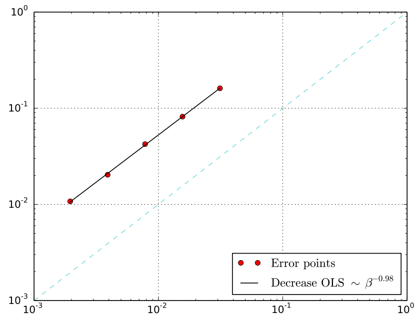
where $\bar{x}_T^{\beta,\Delta t,i}$ and $\bar{Y}_T^{\Delta t,i}$ are independent realisations of the random variables $\bar{x}_T^{\beta,\Delta t}$ and $\bar{Y}_T^{\Delta t}$. For our simulations we consider $N_{\text{MC}} = 10^8$.

Results

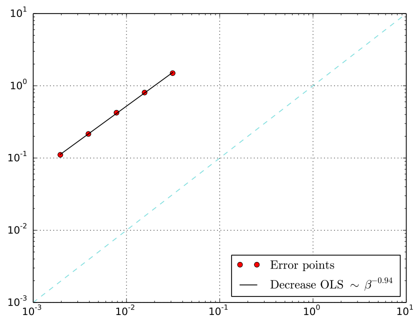
In the log-log plots A.1 we graph the results and also present an ordinary linear regression by using the model $\ln\left(\overline{\text{ErrorSK}}[f]\right) \sim -\alpha \ln(\beta) + \varepsilon$, and the purpose is to estimate the value of α. The thin dashed blue line represents the identity function, presented as a benchmark for the slope of our results.



(a) Case Odd,
$\overline{\text{ErrorSK}}[f] \sim \beta^{-0.95}$



(b) Case Even,
$\overline{\text{ErrorSK}}[f] \sim \beta^{-0.98}$



(c) Case General,
$\overline{\text{ErrorSK}}[f] \sim \beta^{-0.94}$

Figure A.1: Error convergence estimates

In the next table, we present the estimates of the slopes and also the $p-$values for the test: null hypothesis $\alpha = 1$, alternative hypothesis: $\alpha < 1$. In all cases we can see that our theoretical result of error decrease rate bounded in $\beta^{-(1-\epsilon)}$, for any $\varepsilon > 0$, seems to be confirmed.

	Case Odd	Case Even	Case General
OLS Slope Estimation(in $1/\beta$)	0.95	0.98	0.94
p-value	$5.6 \cdot 10^{-3}$	$9.2 \cdot 10^{-2}$	$4.7 \cdot 10^{-3}$

Table A.1: Slope Estimates

We also present in table A.2 the variance and the confidence interval of our estimators. The first line of the table represents the results for the estimator of $f(\bar{Y}_T^{\Delta t})$.

In the Odd and Even cases, the confidence interval is of order 10^{-3}, and since the finest difference in the plots A.1a and A.1b are of order 10^{-2}, the statistical error of Monte-Carlo simulation is not very important. In the General case, the confidence interval is also of order 10^{-3}, while the finest result in the plot A.1c is of order 10^{-1}, so again the statistical error is not significant.

1.1 Switching between the models

In the previous section, we considered a fixed time discretisation step Δt and we varied β. Now, we consider a reversed situation where β is fixed and we vary Δt, in trying to determine if it is more efficient to simulate one model or another for a given target error.

	Case Odd			Case Even			Case General		
	Result	Var	$1/2-$Conf int	Result	Var	$1/2-$Conf int	Result	Var	$1/2-$Conf int
β \ Ref	3.19	20.39	$9.03\cdot10^{-4}$	2.67	7.39	$5.44\cdot10^{-4}$	8.58	73.00	$1.71\cdot10^{-3}$
2^5	2.69	14.47	$7.61\cdot10^{-4}$	2.51	6.61	$5.14\cdot10^{-4}$	7.09	50.83	$1.43\cdot10^{-3}$
2^6	2.93	17.11	$8.27\cdot10^{-4}$	2.58	6.99	$5.29\cdot10^{-4}$	7.78	60.63	$1.56\cdot10^{-3}$
2^7	3.05	18.63	$8.63\cdot10^{-4}$	2.62	7.18	$5.36\cdot10^{-4}$	8.16	66.35	$1.63\cdot10^{-3}$
2^8	3.12	19.53	$8.84\cdot10^{-4}$	2.65	7.29	$5.40\cdot10^{-4}$	8.37	69.60	$1.67\cdot10^{-3}$
2^9	3.16	19.95	$8.93\cdot10^{-4}$	2.66	7.34	$5.42\cdot10^{-4}$	8.47	71.24	$1.69\cdot10^{-3}$

Table A.2: Results Simuation

We plot two types of error

$$\overline{\text{ErrorB}}[f]\colon \Delta t \mapsto \left| \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} \left(f(\bar{x}_T^{\beta,\Delta t^{\text{Ref}},i}) - f(\bar{Y}_T^{\Delta t,i}) \right) \right|$$

and

$$\overline{\text{ErrorL}}[f]\colon \Delta t \mapsto \left| \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} \left(f(\bar{x}_T^{\beta,\Delta t,i}) - f(\bar{x}_T^{\beta,\Delta t^{\text{Ref}},i}) \right) \right|$$

where Δt^{Ref} is a small time step (smaller than the range considered for Δt). For this simulation, we take $N_{\text{MC}} = 10^8$, $\beta = 2^8$ and $\Delta t^{\text{Ref}} = 2^{-11}$. We obtain the following log-log plot

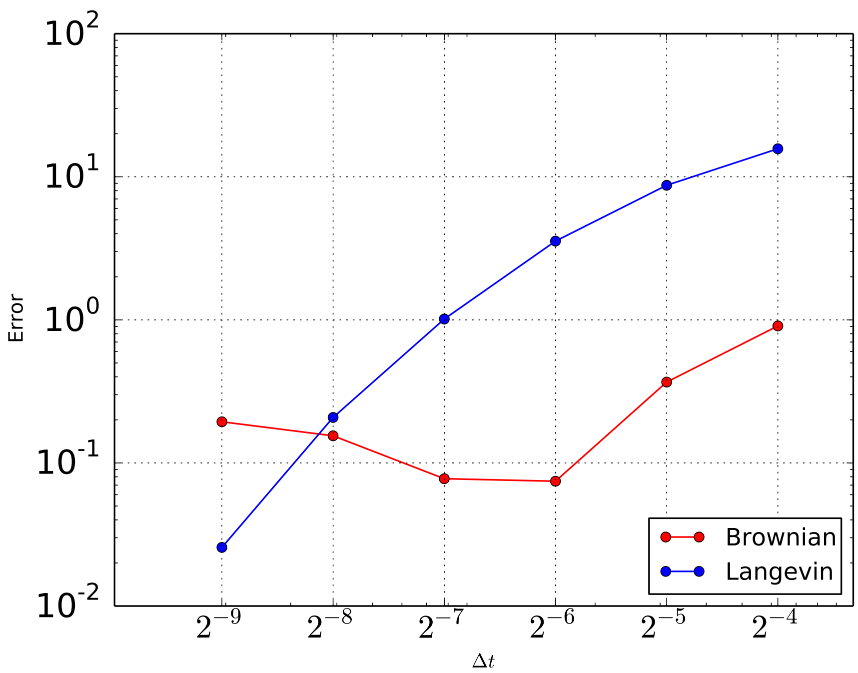


Figure A.2: Plot of $\overline{\text{ErrorB}}[f]$ and $\overline{\text{ErrorL}}[f]$

We recall that we have takes $\beta = 2^8$ so we can notice that for Δt larger than approximately $\frac{1}{\beta}$, that it is more cost-effective to simulate $f(\bar{Y}_T^{\Delta t})$ than to simulate $f(\bar{x}_T^{\beta,\Delta t,i})$. Thus for very large β, if it is too computationally prohibitive to take $\Delta t \leq \frac{1}{\beta}$, it is better to approximate statistics on the position of the Langevin process by the appropriately chosen uniformly elliptic diffusion.

2 Exponential scheme derivation

We consider the process:

$$\begin{cases} x_t = x_0 + \int_0^t u_s \, ds \\ u_t = u_0 - \beta \int_0^t u_s \, ds + \beta\mu t + \beta\sigma W_t \end{cases} \tag{A.1}$$

where μ is a constant. We can explicitly write the solution of this process as:

$$\begin{cases} x_t = x_0 + \frac{u_0}{\beta}(1 - e^{-\beta t}) + \mu \left(t - \frac{1}{\beta}(1 - e^{-\beta t}) \right) + \sigma \int_0^t (1 - e^{-\beta(t-s)}) \, dW_s \\ u_t = u_0 e^{-\beta t} + \mu(1 - e^{-\beta t}) + \beta\sigma \int_0^t e^{-\beta(t-s)} \, dW_s \,, \end{cases} \tag{A.2}$$

meaning that for any $t > 0$ and $(x_0, u_0) \in \mathbb{R}^2$:

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} x_0 + \frac{u_0}{\beta}(1 - e^{-\beta t}) + \mu \left(t - \frac{1}{\beta}(1 - e^{-\beta t}) \right) \\ u_0 e^{-\beta t} + \mu(1 - e^{-\beta t}) \end{bmatrix}, \Sigma \right) \tag{A.3}$$

where Σ is the positive definite covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma^2 \left(t - \frac{2}{\beta}(1 - e^{-\beta t}) + \frac{1}{2\beta}(1 - e^{-2\beta t}) \right) & \cdot \\ \frac{\sigma^2}{2}(1 - e^{-\beta t})^2 & \beta \frac{\sigma^2}{2}(1 - e^{-2\beta t}) \end{bmatrix} = \begin{bmatrix} \Sigma_{xx}^2(t) & \rho\Sigma_{xx}\Sigma_{uu}(t) \\ \rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t) & \Sigma_{uu}^2(t) \end{bmatrix} . \tag{A.4}$$

For $\mu \equiv 0$, we have that

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} x_0 + \frac{u_0}{\beta}(1 - e^{-\beta t}) \\ u_0 e^{-\beta t} \end{bmatrix}, \Sigma \right) . \tag{A.5}$$

3 Bounds on the various moments of the Ornstein-Uhlenbeck process with constant drift

Let $(u_t)_{t \geq 0}$ verify equation (A.1) and $p \geq 1$, then:

$$\begin{aligned} du_t^p &= p u_t^{p-1} \, du_t + \frac{1}{2} p(p-1) u_t^{p-2} \beta^2 \sigma^2 \, dt \\ &= -\beta p u_t^p \, dt + p\beta\mu u_t^{p-1} \, dt + \frac{1}{2} p(p-1) \beta^2 \sigma^2 u_t^{p-2} \, dt + \beta\sigma p u_t^{p-1} \, dW_t \,. \end{aligned}$$

We have that:

$$\begin{aligned} u_t^p = u_0^p e^{-p\beta t} &+ p\beta\mu \int_0^t e^{-p\beta(t-s)} u_s^{p-1} \, ds + \frac{1}{2} p(p-1) \beta^2 \sigma^2 \int_0^t e^{-p\beta(t-s)} u_s^{p-2} \, ds \\ &+ \beta\sigma p \int_0^t e^{-p\beta(t-s)} u_s^{p-1} \, dW_s \,. \end{aligned} \tag{A.1}$$

By taking the expectation, we obtain that:

$$\mathbb{E}u_t^p = u_0^p e^{-p\beta t} + p\beta\mu \int_0^t e^{-p\beta(t-s)}\mathbb{E}\left[u_s^{p-1}\right]\,ds + \frac{1}{2}p(p-1)\beta^2\sigma^2 \int_0^t e^{-p\beta(t-s)}\mathbb{E}\left[u_s^{p-2}\right]\,ds\,. \tag{A.2}$$

For $p=1$, we have that:

$$\mathbb{E}u_t = u_0 e^{-\beta t} + \mu(1-e^{-\beta t}) \tag{A.3}$$

while for $p=2$:

$$\begin{aligned}\mathbb{E}u_t^2 &= u_0^2 e^{-2\beta t} + 2\beta\mu \int_0^t e^{-2\beta(t-s)}\left(u_0 e^{-\beta s} + \mu(1-e^{-\beta s})\right)\,ds \\ &\quad + \beta^2\sigma^2 \int_0^t e^{-2\beta(t-s)}\,ds \\ &= u_0^2 e^{-2\beta t} + 2u_0\mu e^{-\beta t}(1-e^{-\beta t}) + \mu^2(1-e^{-\beta t})^2 + \frac{\beta\sigma^2}{2}(1-e^{-2\beta t})\end{aligned} \tag{A.4}$$

and we have the bound for the second moment:

$$\mathbb{E}u_t^2 \le C_\mu \beta \tag{A.5}$$

while for the third moment:

$$\left|\mathbb{E}u_t^3\right| \le u_0^3 e^{-3\beta t} + 3kC_\mu\beta \int_0^t e^{-3\beta(t-s)}\,ds + 3\beta^2\sigma^2 C_\mu \int_0^t e^{-3\beta(t-s)}\,ds \le C_\mu\beta\,. \tag{A.6}$$

Finally:

$$\mathbb{E}u_t^4 \le u_0^4 e^{-4\beta t} + 4C_\mu\beta^2 \int_0^t e^{-4\beta(t-s)}\,ds + C_\mu 6\beta^3\sigma^2 \int_0^t e^{-4\beta(t-s)}\,ds \le C_\mu\beta^2\,. \tag{A.7}$$

We can conclude that:

$$\mathbb{E}\left|u_t^3\right| \le \sqrt{\mathbb{E}u_t^2}\sqrt{\mathbb{E}u_t^4} \le C_\mu\beta\sqrt{\beta}\,. \tag{A.8}$$

Now, we consider a smooth enough function g, then:

$$\begin{aligned}d(g(Y_t)u_t^p) &= u_t^p\,dg(Y_t) + g(Y_t)\,du_t^p + d\langle g(Y_\cdot), u_\cdot^p\rangle_t = u_t^p g'(Y_t)\,dY_t + \frac{\sigma^2}{2}u_t^p g''(Y_t)\,dt \\ &\quad - \beta p g(Y_t)u_t^p\,dt + p\beta\mu u_t^{p-1}g(Y_t)\,dt + \frac{1}{2}p(p-1)\beta^2\sigma^2 u_t^{p-2}g(Y_t)\,dt \\ &\quad + \beta\sigma p u_t^{p-1}g(Y_t)\,dW_t + \beta\sigma^2 p u_t^{p-1}g'(Y_t)\,dt\,.\end{aligned} \tag{A.9}$$

And by a similar procedure as the Ornstein Uhlenbeck process, we obtain:

$$\begin{aligned}d(e^{p\beta t}g(Y_t)u_t^p) &= u_t^p e^{p\beta t}g'(Y_t)\,dY_t + \frac{\sigma^2}{2}e^{p\beta t}u_t^p g''(Y_t)\,dt \\ &\quad + p\beta\mu e^{p\beta t}u_t^{p-1}g(Y_t)\,dt + \frac{1}{2}p(p-1)\beta^2\sigma^2 e^{p\beta t}u_t^{p-2}g(Y_t)\,dt \\ &\quad + \beta\sigma p e^{p\beta t}u_t^{p-1}g(Y_t)\,dW_t + \beta\sigma^2 p e^{p\beta t}u_t^{p-1}g'(Y_t)\,dt\end{aligned} \tag{A.10}$$

obtaining that:

$$\begin{aligned}g(Y_t)u_t^p &= e^{-p\beta t}g(x_0)u_0^p + \int_0^t e^{-p\beta(t-r)}u_r^p g'(Y_r)\,dY_r + \frac{\sigma^2}{2}\int_0^t e^{-p\beta(t-r)}u_r^p g''(Y_r)\,dr \\ &\quad + p\beta\mu \int_0^t e^{-p\beta(t-r)}u_r^{p-1}g(Y_r)\,dr + \frac{p(p-1)\sigma^2\beta^2}{2}\int_0^t e^{-p\beta(t-r)}u_r^{p-2}g(Y_r)\,dr \\ &\quad + \beta\sigma p \int_0^t e^{-p\beta(t-r)}u_r^{p-1}g(Y_r)\,dW_r + \beta\sigma^2 p \int_0^t e^{-p\beta(t-r)}u_r^{p-1}g'(Y_r)\,dr\,.\end{aligned} \tag{A.11}$$

We can integrate to obtain:

$$\begin{aligned}
\int_0^t g(Y_s)u_s^p\,ds &= \frac{1-e^{-p\beta t}}{p\beta}g(x_0)u_0^p + \mu\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^p g'(Y_r)\,drds \\
&+ \sigma\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^p g'(Y_r)\,dW_r ds + \frac{\sigma^2}{2}\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^p g''(Y_r)\,drds \\
&+ p\beta\mu\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-1}g(Y_r)\,drds + \frac{p(p-1)\sigma^2\beta^2}{2}\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-2}g(Y_r)\,drds \\
&+ \beta\sigma p\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-1}g(Y_r)\,dW_r ds + \beta\sigma^2 p\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-1}g'(Y_r)\,drds
\end{aligned} \tag{A.12}$$

and an integration by parts results in:

$$\begin{aligned}
\int_0^t g(Y_s)u_s^p\,ds &= \frac{1-e^{-p\beta t}}{p\beta}g(x_0)u_0^p - \frac{\mu}{p\beta}\int_0^t e^{-p\beta(t-r)}u_r^p g'(Y_r)\,dr + \frac{\mu}{p\beta}\int_0^t u_s^p g'(Y_s)\,ds \\
&+ \sigma\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^p g'(Y_r)\,dW_r ds - \frac{\sigma^2}{2p\beta}\int_0^t e^{-p\beta(t-r)}u_r^p g''(Y_r)\,dr + \frac{\sigma^2}{2p\beta}\int_0^t u_r^p g''(Y_r)\,dr \\
&- \mu\int_0^t e^{-p\beta(t-r)}u_r^{p-1}g(Y_r)\,dr + \mu\int_0^t u_r^{p-1}g(Y_r)\,dr - \frac{(p-1)\sigma^2\beta}{2}\int_0^t e^{-p\beta(t-r)}u_r^{p-2}g(Y_r)\,dr \\
&+ \frac{(p-1)\sigma^2\beta}{2}\int_0^t u_r^{p-2}g(Y_r)\,dr + \beta\sigma p\int_0^t\int_0^s e^{-p\beta(s-r)}u_r^{p-1}g(Y_r)\,dW_r ds \\
&- \sigma^2\int_0^t e^{-p\beta(t-r)}u_r^{p-1}g'(Y_r)\,dr + \sigma^2\int_0^t u_r^{p-1}g'(Y_r)\,dr\,.
\end{aligned} \tag{A.13}$$

4 Various useful calculations

We consider:

$$\begin{aligned}
\beta C(\beta,\sigma,t) &= \frac{1}{\sqrt{2\pi}}\frac{\left|(1-e^{-\beta t})\Sigma_{uu}(t) - \beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)\right|}{(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)} \\
&+ \frac{1}{\sqrt{2\pi}}\frac{\left|\beta e^{-\beta t}\Sigma_{xx}(t) - (1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)\right|}{(1-\rho^2(t))\Sigma_{xx}(t)\Sigma_{uu}(t)}
\end{aligned} \tag{A.1}$$

and have that:

$$\begin{aligned}
\beta C(\beta,\sigma,t) &\le \frac{1}{\sqrt{2\pi}}\frac{\left((1-e^{-\beta t})\Sigma_{uu}(t) + \beta e^{-\beta t}\rho(t)\Sigma_{xx}(t)\right)\Sigma_{xx}(t)\Sigma_{uu}(t)}{\Sigma_{xx}^2(t)\Sigma_{uu}^2(t) - (\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t))^2} \\
&+ \frac{1}{\sqrt{2\pi}}\frac{\left(\beta e^{-\beta t}\Sigma_{xx}(t) + (1-e^{-\beta t})\rho(t)\Sigma_{uu}(t)\right)\Sigma_{xx}(t)\Sigma_{uu}(t)}{\Sigma_{xx}^2(t)\Sigma_{uu}^2(t) - (\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t))^2}\,.
\end{aligned} \tag{A.2}$$

It can be seen that all the terms have as denominator:

$$\begin{aligned}
\Sigma_{xx}^2(t)\Sigma_{uu}^2(t) - (\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t))^2 &= \frac{\sigma^4}{2}\beta\left(t - \frac{2}{\beta}(1-e^{-\beta t}) + \frac{1}{2\beta}(1-e^{-2\beta t})\right)(1-e^{-2\beta t}) \\
&- \frac{\sigma^4}{4}\left(1-e^{-\beta t}\right)^4 = \sigma^4 h(\beta t)
\end{aligned} \tag{A.3}$$

where

$$h\colon z \mapsto \frac{1}{2}(1-e^{-2z})\left(z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})\right)-\left(1-e^{-z}-\frac{1}{2}(1-e^{-2z})\right)^2 \tag{A.4}$$

which we rewrite as

$$h(z)=(1-e^{-z})\left(\frac{z}{2}(1+e^{-z})-(1-e^{-z})\right). \tag{A.5}$$

We can notice that:

$$h(z)\geq \frac{1}{2}(1-e^{-2z})(z-2)-\frac{1}{4} \tag{A.6}$$

and for $z \geq 4 > \frac{\ln 2}{2}$:

$$h(z)\geq \frac{1}{4}(z-3)\,. \tag{A.7}$$

From the definition of the different terms of the covariance matrix in (A.4), we have that

$$\frac{(1-e^{-\beta t})\Sigma_{xx}(t)\Sigma_{uu}^2(t)}{\sigma^4 h(\beta t)}=\frac{1}{\sigma}\frac{\sqrt{\beta}}{2}I_{\mathbf{I.1}}(\beta t) \tag{A.8}$$

where for any $z>0$:

$$I_{\mathbf{I.1}}\colon z\mapsto \frac{1}{h(z)}(1-e^{-z})(1-e^{-2z})\sqrt{z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})}$$

so :

$$I_{\mathbf{I.1}}\colon z\mapsto (1-e^{-2z})\frac{\sqrt{z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})}}{\frac{z}{2}(1+e^{-z})-(1-e^{-z})}\,. \tag{A.9}$$

We also have

$$\frac{\beta e^{-\beta t}\Sigma_{xx}(t)\left(\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t)\right)}{\sigma^4 h(\beta t)}=\frac{1}{\sigma}\sqrt{\beta}I_{\mathbf{I.2}}(\beta t) \tag{A.10}$$

where for any $z>0$, we have that

$$I_{\mathbf{I.2}}\colon z\mapsto \frac{1}{2}\frac{e^{-z}}{h(z)}\sqrt{z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})}\left(1-e^{-z}\right)^2.$$

thus

$$I_{\mathbf{I.2}}\colon z\mapsto \frac{1}{2}e^{-z}\left(1-e^{-z}\right)\frac{\sqrt{z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})}}{\frac{z}{2}(1+e^{-z})-(1-e^{-z})}\,. \tag{A.11}$$

The third term is

$$\frac{\beta e^{-\beta t}\Sigma_{xx}^2(t)\Sigma_{uu}(t)}{\sigma^4 h(\beta t)}=\frac{1}{\sigma}\sqrt{\beta}I_{\mathbf{II.1}}(\beta t) \tag{A.12}$$

where

$$I_{\mathbf{II.1}}\colon z\mapsto \frac{1}{\sqrt{2}}\frac{e^{-z}}{h(z)}\left(z-2(1-e^{-z})+\frac{1}{2}(1-e^{-2z})\right)\sqrt{1-e^{-2z}}\,. \tag{A.13}$$

Finally, the fourth term is

$$\frac{(1-e^{-\beta t})\Sigma_{uu}(t)\left(\rho(t)\Sigma_{xx}(t)\Sigma_{uu}(t)\right)}{\sigma^4 h(\beta t)}=\frac{1}{\sigma}\sqrt{\beta}I_{\mathbf{II.2}}(\beta t) \tag{A.14}$$

where:

$$I_{\mathbf{II.2}}\colon z \mapsto \frac{1}{2\sqrt{2}} \frac{(1-e^{-z})^3}{h(z)} \sqrt{1-e^{-2z}}\,.$$

Lemma 4.1. *We have that there exists a constant $C > 0$ such that for any $z > 0$*

(i) $I_{\mathbf{I.1}}(z) \leq \frac{C}{\sqrt{z}}$,

(ii) $I_{\mathbf{I.2}}(z) \leq \frac{C}{\sqrt{z}}$,

(iii) $I_{\mathbf{II.1}}(z) \leq \frac{C}{\sqrt{z}}$,

(iv) $I_{\mathbf{II.2}}(z) \leq \frac{C}{\sqrt{z}}$.

Proof. **Item *(i)*** We have for z close to 0, that

$$I_{\mathbf{I.1}}(z) = \frac{4\sqrt{3}}{\sqrt{z}} - \frac{7\sqrt{3}}{2}\sqrt{z} + o(\sqrt{z}) \tag{A.15}$$

so there is a constant $\varepsilon_{\mathbf{I.1}} > 0$ such that for any $z \in (0, \varepsilon_{\mathbf{I.1}})$, we have that

$$I_{\mathbf{I.1}}(z) \leq \frac{4\sqrt{3}}{\sqrt{z}}$$

since $I_{\mathbf{I.1}}(z) - \frac{4\sqrt{3}}{\sqrt{z}} = -\frac{7\sqrt{3}}{2}\sqrt{z} + o(\sqrt{z})$ and $\varepsilon_{\mathbf{I.1}}$ is chosen such that on $(0, \varepsilon_{\mathbf{I.1}})$ we have that $\frac{7\sqrt{3}}{2}\sqrt{z} \geq o(\sqrt{z})$.

Also, we notice that

$$I_{\mathbf{I.1}}(z) = (1-e^{-2z}) \frac{\sqrt{1 - \frac{2}{z}(1-e^{-z}) + \frac{1}{2z}(1-e^{-2z})}}{\frac{\sqrt{z}}{2}(1+e^{-z}) - \frac{1}{\sqrt{z}}(1-e^{-z})}$$

There exists $M_{\mathbf{I.1}} > \varepsilon_{\mathbf{I.1}} > 0$ such that for any $z > M_{\mathbf{I.1}}$, $1-e^{-z} \leq \frac{z}{3}$, meaning that $-\frac{1}{\sqrt{z}}(1-e^{-z}) \geq -\frac{\sqrt{z}}{3}$, thus

$$\frac{\sqrt{z}}{2}(1+e^{-z}) - \frac{1}{\sqrt{z}}(1-e^{-z}) \geq \frac{\sqrt{z}}{2} - \frac{\sqrt{z}}{3} = \frac{\sqrt{z}}{6} > 0$$

so for any $z > M_{\mathbf{I.1}} \vee 1$, we have that

$$I_{\mathbf{I.1}}(z) \leq \frac{\sqrt{1+\frac{1}{2}}}{\frac{\sqrt{z}}{6}} \leq \frac{6\sqrt{2}}{\sqrt{z}}\,.$$

On the compact interval $[\varepsilon_{\mathbf{I.1}}, M_{\mathbf{I.1}} \vee 1]$, the functions $z \mapsto I_{\mathbf{I.1}}(z)$ and $z \mapsto \frac{1}{\sqrt{z}}$ are continuous and strictly positive, so by taking $C_{\mathbf{I.1}} = \sqrt{M_{\mathbf{I.1}} \vee 1}\max_{z\in[\varepsilon_{\mathbf{I.1}}, M_{\mathbf{I.1}}\vee 1]}(I_{\mathbf{I.1}}(z)) > 0$ we have for any $z \in [\varepsilon_{\mathbf{I.1}}, M_{\mathbf{I.1}} \vee 1]$

$$\frac{1}{C_{\mathbf{I.1}}} I_{\mathbf{I.1}}(z) \leq \frac{1}{\sqrt{M_{\mathbf{I.1}} \vee 1}} \frac{I_{\mathbf{I.1}}(z)}{\max_{z\in[\varepsilon_{\mathbf{I.1}}, M_{\mathbf{I.1}}\vee 1]}(I_{\mathbf{I.1}}(z))} \leq \frac{1}{\sqrt{M_{\mathbf{I.1}} \vee 1}} \leq \frac{1}{\sqrt{z}} \tag{A.16}$$

thus

$$I_{\mathbf{I.1}}(z) \leq \frac{C_{\mathbf{I.1}}}{\sqrt{z}}\,.$$

We conclude that for any $z > 0$:

$$\begin{aligned}
I_{\mathbf{I.1}}(z) &\leq \frac{4\sqrt{3}}{\sqrt{z}}\mathbb{1}_{z\in(0,\varepsilon_{\mathbf{I.1}})} + \frac{C_{\mathbf{I.1}}}{\sqrt{z}}\mathbb{1}_{z\in[\varepsilon_{\mathbf{I.1}},M_{\mathbf{I.1}}\vee 1]} + \frac{6\sqrt{2}}{\sqrt{z}}\mathbb{1}_{z\in(M_{\mathbf{I.1}}\vee 1,+\infty)} \\
&\leq \frac{4\sqrt{3}}{\sqrt{z}}\mathbb{1}_{z\in(0,+\infty)} + \frac{C_{\mathbf{I.1}}}{\sqrt{z}}\mathbb{1}_{z\in(0,+\infty)} + \frac{6\sqrt{2}}{\sqrt{z}}\mathbb{1}_{z\in(0,+\infty)} \\
&\leq \frac{1}{\sqrt{z}}\mathbb{1}_{z\in(0,+\infty)}\left(4\sqrt{3} + C_{\mathbf{I.1}} + 6\sqrt{2}\right)
\end{aligned} \tag{A.17}$$

Item *(ii)* For any $z > 0$, we have that:

$$e^{-z}(1-e^{-z}) \leq 1-e^{-z} \leq 1-e^{-2z}$$

thus it is easy to see that:

$$I_{\mathbf{I.2}}(z) \leq \frac{1}{2}I_{\mathbf{I.1}}(z)$$

and we conclude using the previous result *(i)*.

Item *(iii)* We introduce the function

$$J_{\mathbf{II.1}} \colon z \mapsto \frac{1}{\sqrt{2}}\frac{1}{h(z)}\left(z - 2(1-e^{-z}) + \frac{1}{2}(1-e^{-2z})\right)\sqrt{1-e^{-2z}}$$

and it can be easily seen that $I_{\mathbf{II.1}}(z) = e^{-z}J_{\mathbf{II.1}}(z)$.

We have that for z close to 0

$$J_{\mathbf{II.1}}(z) = \frac{4}{\sqrt{z}} - \sqrt{z} + o(\sqrt{z})$$

so there exists $\varepsilon_{\mathbf{II.1}} > 0$, such that for any $z \in (0, \varepsilon_{\mathbf{II.1}})$

$$J_{\mathbf{II.1}}(z) \leq \frac{4}{\sqrt{z}}$$

as $J_{\mathbf{II.1}}(z) - \frac{4}{\sqrt{z}} = -\sqrt{z} + o(\sqrt{z})$ and on $(0, \varepsilon_{\mathbf{II.1}})$, $\sqrt{z} \geq o(\sqrt{z})$.

Rewriting $J_{\mathbf{II.1}}$ as:

$$\begin{aligned}
J_{\mathbf{II.1}}(z) &= \frac{1}{\sqrt{2}}\sqrt{\frac{1+e^{-z}}{1-e^{-z}}}\frac{z - 2(1-e^{-z}) + \frac{1}{2}(1-e^{-2z})}{\frac{z}{2}(1+e^{-z}) - (1-e^{-z})} \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{1+e^{-z}}{1-e^{-z}}}\frac{1 - \frac{2}{z}(1-e^{-z}) + \frac{1}{2z}(1-e^{-2z})}{\frac{1}{2}(1+e^{-z}) - \frac{1}{z}(1-e^{-z})}
\end{aligned}$$

we notice that $J_{\mathbf{II.1}}(z) \xrightarrow{z\to+\infty} \sqrt{2}$, so there exists $M_{\mathbf{II.1}} > \varepsilon_{\mathbf{II.1}} > 0$ such that for any $z > M_{\mathbf{II.1}}$, $J_{\mathbf{II.1}}(z) \leq 2$. Also on the interval $[\varepsilon_{\mathbf{II.1}}, M_{\mathbf{II.1}}]$, the function $J_{\mathbf{II.1}}(z)$ is continuous and strictly positive so there exists $C_{\mathbf{II.1}} > 0$ such that for any $z \in [\varepsilon_{\mathbf{II.1}}, M_{\mathbf{II.1}}]$, $J_{\mathbf{II.1}}(z) \leq C_{\mathbf{II.1}}$. We obtain, then that for any $z > 0$:

$$\begin{aligned}
I_{\mathbf{II.1}}(z) = e^{-z}J_{\mathbf{II.1}}(z) &\leq e^{-z}\left(\frac{4}{\sqrt{z}} + C_{\mathbf{II.1}} + 2\right) \leq \frac{4}{\sqrt{z}} + e^{-z}(C_{\mathbf{II.1}} + 2) \\
&\leq \frac{4}{\sqrt{z}} + \frac{1}{\sqrt{z}}(C_{\mathbf{II.1}} + 2) \\
&\leq \frac{1}{\sqrt{z}}(6 + C_{\mathbf{II.1}})
\end{aligned} \tag{A.18}$$

Item *(iv)* We have for z close to 0, that

$$I_{\mathbf{II.2}}(z) = \frac{6}{\sqrt{z}} - 6\sqrt{z} + o(\sqrt{z})$$

so by similar arguments to *(i)*, we obtain a similar bound.

Taking the maximum over all the constants allows us to obtain the result presented in the Lemma. ■

Corollary 4.2. *There exists a constant $K > 0$ such that $\beta C(\beta, \sigma, t) \leq \frac{K}{\sqrt{t}}$, where the function C is defined in* (A.1).

Proof. By the inequality (A.2), we have that:

$$\beta C(\beta, \sigma, t) \leq \frac{1}{\sqrt{2\pi}\sigma}\sqrt{\beta}\left(I_{\mathbf{I.1}}(\beta t) + I_{\mathbf{I.2}}(\beta t) + I_{\mathbf{II.1}}(\beta t) + I_{\mathbf{II.2}}(\beta t)\right)$$

and by the Lemma 4.1, we obtain that

$$\beta C(\beta, \sigma, t) \leq \frac{4C}{\sqrt{2\pi}\sigma}\sqrt{\beta}\frac{1}{\sqrt{\beta t}} \leq \frac{4C}{\sqrt{2\pi}\sigma}\frac{1}{\sqrt{t}}\,.$$

■

4.1 Controls for different functions and integrals

We now consider the following function

$$\frac{1 - e^{-\beta t}}{\Sigma_{xx}(t)} = \sqrt{\beta}\frac{1}{\sigma}K(\beta t)$$

where $K\colon (0, +\infty) \mapsto \mathbb{R}$ defined as

$$K\colon z \mapsto \frac{1 - e^{-z}}{\sqrt{z - 2(1 - e^{-z}) + \frac{1}{2}(1 - e^{-2z})}}\,.$$

For z close to 0, we have that

$$K(z) = \frac{\sqrt{3}}{\sqrt{z}} - \frac{\sqrt{3z}}{8} + o(\sqrt{z})$$

so

$$K(z) \leq \frac{\sqrt{3}}{\sqrt{z}}$$

for small enough z, as $K(z) - \frac{\sqrt{3}}{\sqrt{z}} = -\frac{\sqrt{3z}}{8} + o(\sqrt{z})$.

There exists a large enough M such that for any $z > M$, we have that $2(1-e^{-z})+\frac{1}{2}(1-e^{-2z}) \leq \frac{z}{2}$, therefore for $z > M$, we have that $K(z) \leq \frac{\sqrt{2}}{\sqrt{z}}$. By using similar arguments to the ones presented in the proof of the previous lemma, we have that there exists a constant C, such that for any $z > 0$,

$$K(z) \leq \frac{C}{\sqrt{z}}\,.$$

We therefore obtain that

$$\frac{1-e^{-\beta t}}{\Sigma_{xx}(t)} \leq C\frac{1}{\sigma}\sqrt{\beta}\frac{1}{\sqrt{\beta t}} \leq \frac{C}{\sigma}\frac{1}{\sqrt{t}}\,. \tag{A.19}$$

Also we have the integral:

$$\int_0^t \frac{(1-e^{-\beta s})e^{-\beta s}}{\Sigma_{xx}(s)}\,ds = \frac{1}{\sigma\sqrt{\beta}}\int_0^{\beta t} K(z)e^{-z}\,dz\,. \tag{A.20}$$

For β sufficiently large, $\beta > \frac{3}{t}$, we have that $\int_0^3 K(z)e^{-z}\,dz$ is bounded. Also:

$$\int_3^{\beta t} K(z)e^{-z}\,dz \leq \int_3^{\beta t}\frac{e^{-z}}{\sqrt{z-2}}\,dz = \frac{\sqrt{\pi}}{e^2}\left(\text{erf}\left(\sqrt{\beta t-2}\right)-\text{erf}(1)\right) \leq \frac{\sqrt{\pi}}{e^2}\,. \tag{A.21}$$

Thus the integral:

$$\int_0^t \frac{(1-e^{-\beta s})e^{-\beta s}}{\Sigma_{xx}(s)}\,ds \leq C_\sigma\frac{1}{\sqrt{\beta}} \tag{A.22}$$

where $C_\sigma > 0$ depends on σ.

We also have the following integral, decomposed for $\beta \geq \frac{1}{t}$:

$$\begin{aligned}\int_0^t \frac{(1-e^{-\beta s})^2}{s}\,ds &= \int_0^{\beta t}\frac{(1-e^{-z})^2}{z}\,dz \leq \int_0^1 \frac{(1-e^{-z})^2}{z}\,dz + \int_1^{\beta t}\frac{1}{z}\,dz \\ &\leq C+\ln(\beta t) \leq C_t + \ln(\beta)\end{aligned} \tag{A.23}$$

where $C_t = C + \ln(t) = 0.273936.. + \ln(t)$.

We also present a version of Gronwall's inequality.

Theorem 4.3 ([Ye *et al.*, 2007])**.** *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$(some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$(constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)\,ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds, \quad 0 \leq t < T\,.$$

Remark 4.4. *Assume that for any $t \geq 0$, $g(t) \equiv K$, where K is a positive constant and $\alpha = \frac{1}{2}$. Then we have that*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1} &= \sum_{n=1}^{\infty}\frac{K^n\Gamma^n\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}(t-s)^{\frac{n}{2}-1} \\ &= \frac{K}{\sqrt{t-s}} + \pi K e^{\pi K^2(t-s)} + \pi K e^{\pi K^2(t-s)}\,\text{erf}(K\sqrt{\pi(t-s)})\end{aligned} \tag{A.24}$$

Furthermore, if the function a is constant, (for all $t \geq 0$, $a(t) \equiv a$, then if

$$u(t) \leq a + \int_0^t \frac{u(s)}{\sqrt{t-s}}\,ds$$

then

$$u(t) \leq a e^{\pi K^2 t}\left(1+\text{erf}(K\sqrt{\pi t})\right) \leq 2a e^{\pi K^2 t} \tag{A.25}$$

Lemma 4.5. *There exists $C_{\beta,\mu_0,\mu,\sigma,T} > 0$ that depends on β, μ_0, μ, $T > 0$, such that the solution ρ of the mild equation* (2.28) *verifies*

(i) $\sup_{(t,x,u)\in[0,T]\times\mathbb{R}^2} \rho(t,x,u) \leq C_{\beta,\mu_0,\mu,\sigma,T}$,

(ii) $\sup_{(t,x,u)\in[0,T]\times\mathbb{R}^2} |\partial_x\rho(t,x,u)| \leq C_{\beta,\mu_0,\mu,\sigma,T}$.

Proof. **Item *(i)*** We rewrite (2.28) as

$$\begin{aligned}\rho(t,x,u) = &\int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\mu_0(y,v)\,dydv\\ &+ \beta\int_0^t\int_{\mathbb{R}^2} \partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\rho(s,y,v)\mu(y)\,dydvds\end{aligned}\tag{A.26}$$

thus

$$\begin{aligned}\rho(t,x,u) \leq &\sup_{(y,v)\in\mathbb{R}^2}\mu_0\int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\,dydv\\ &+ \beta\,\|\mu\|_{L^\infty(\mathbb{R})}\int_0^t\left(\sup_{(y,v)\in\mathbb{R}^2}\rho(s,y,v)\right)\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)|\,dsdydv\,.\end{aligned}\tag{A.27}$$

The equation (2.65) gives the value of $\partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)$. It can also be seen that

$$\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t;y,v;x,u)|\,dxdu = e^{\beta t}\int_{\mathbb{R}^2}|\partial_v\Gamma_{\mathrm{OU}}(t;y,v;x,u)|\,dydv \leq e^{\beta t}C(\beta,\sigma,t)\tag{A.28}$$

where the bound $C(\beta,\sigma,t)$ is defined in (2.67). By Corollary 4.2, we have that there exists $K > 0$ such that $\beta C(\beta,\sigma,t) \leq \frac{K}{\sqrt{t}}$ thus we have that for any $t\in[0,T]$:

$$\begin{aligned}\rho(t,x,u) &\leq \sup_{(y,v)\in\mathbb{R}^2}\mu_0 + K\,\|\mu\|_{L^\infty(\mathbb{R})}\int_0^t\left(\sup_{(y,v)\in\mathbb{R}^2}\rho(s,y,v)\right)\frac{e^{\beta(t-s)}}{\sqrt{t-s}}\,ds\\ &\leq \sup_{(y,v)\in\mathbb{R}^2}\mu_0 + K\,\|\mu\|_{L^\infty(\mathbb{R})}\,e^{\beta T}\int_0^t\left(\sup_{(y,v)\in\mathbb{R}^2}\rho(s,y,v)\right)\frac{1}{\sqrt{t-s}}\,ds\end{aligned}\tag{A.29}$$

by taking the supremum over $(x,u)\in\mathbb{R}^2$ and applying Gronwall's inequality as in the Remark 4.4, we have that for any $t\in[0,T]$

$$\sup_{(x,u)\in\mathbb{R}^2}\rho(t,x,u) \leq 2\sup_{(y,v)\in\mathbb{R}^2}\mu_0\exp\left(\pi K^2\,\|\mu\|^2_{L^\infty(\mathbb{R})}\,e^{2\beta T}t\right)\tag{A.30}$$

thus obtaining the required result since exp is an increasing function.

Item *(ii)* We differentiate (2.28) to obtain

$$\begin{aligned}\partial_x\rho(t,x,u) = &\int_{\mathbb{R}\times\mathbb{R}} \partial_x\Gamma_{\mathrm{OU}}(t;y,v;x,u)\mu_0(y,v)\,dydv\\ &+ \beta\int_0^t\int_{\mathbb{R}^2} \partial_v\partial_x\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\rho(s,y,v)\mu(y)\,dydvds\,.\end{aligned}\tag{A.31}$$

It is straightforward to see that $\partial_x\Gamma_{\mathrm{OU}}(t-s;y,v;x,u) = -\partial_y\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)$ and we can apply an integration by parts to obtain

$$\begin{aligned}\partial_x\rho(t,x,u) = &\int_{\mathbb{R}\times\mathbb{R}} \Gamma_{\mathrm{OU}}(t;y,v;x,u)\partial_y\mu_0(y,v)\,dydv\\ &+ \beta\int_0^t\int_{\mathbb{R}^2} \partial_v\Gamma_{\mathrm{OU}}(t-s;y,v;x,u)\partial_y\left(\rho(s,y,v)\mu(y)\right)\,dydvds\,.\end{aligned}\tag{A.32}$$

The boundary terms of the integration by parts are null since $\partial_v \Gamma_{\text{OU}}(t-s;y,v;x,u)$ and $\Gamma_{\text{OU}}(t-s;y,v;x,u)$ go to 0 as $|y| \to \infty$, μ_0 vanishes at infinity while ρ is bounded in y as shown previously. We can see that we obtain an equation that has a similar form to the mild equation verified by ρ, so we can apply the same arguments as in *(i)* to obtain the desired result.

■

Bibliography

[Abdulle and Li, 2008] Assyr Abdulle and Tiejun Li. *S-ROCK methods for stiff Itô SDEs*. 6:845–868, 2008. (Cited on page 113.)

[Bally and Talay, 1996] V. Bally and D. Talay. *The law of the Euler scheme for stochastic differential equations*. Probability Theory and Related Fields, 104(1):43–60, 1996. (Cited on page 92.)

[Bernardin *et al.*, 2010] Frédéric Bernardin, Mireille Bossy, Claire Chauvin, Jean Francois Jabir, and Antoine Rousseau. *Stochastic Lagrangian Method for Downscaling Problems in Computational Fluid Dynamics*. ESAIM: Mathematical Modelling and Numerical Analysis, 44(5):885–920, 2010. (Cited on pages 10 and 16.)

[Bertoin, 2007] Jean Bertoin. *Reflecting a Langevin process at an absorbing boundary*. Ann. Probab., 35(6):2021–2037, 2007. (Cited on page 14.)

[Blount and Bose, 2000] Douglas Blount and Amit Bose. *Fourier Analysis Applied to Super Stable and Related Processes*. In *Proceedings on the International Conference on Stochastic Models*, volume 26, pages 25–40. 2000. (Cited on page 139.)

[Bossy *et al.*, 2016] Mireille Bossy, Jose Espina, Jacques Morice, Cristian Paris, and Antoine Rousseau. *Modeling the wind circulation around mills with a Lagrangian stochastic approach*. SMAI-Journal of computational mathematics, 2(2):177–214, 2016. (Cited on page 16.)

[Bossy *et al.*, 2004] Mireille Bossy, Emmanuel Gobet, and Denis Talay. *Symmetrized Euler scheme for an efficient approximation of reflected diffusions*. J. Appl. Probab., 41(3):877–889, 2004. (Cited on pages 14, 92, and 179.)

[Bossy and Jabir, 2011] Mireille Bossy and Jean-François Jabir. *On confined McKean Langevin processes satisfying the mean no-permeability boundary condition*. Stochastic Processes and their Applications, 121(12):2751 – 2775, 2011. (Cited on pages 14, 15, 23, 24, 52, 70, 89, 109, 164, and 165.)

[Bossy and Jabir, 2015] Mireille Bossy and Jean-François Jabir. *Lagrangian stochastic models with specular boundary condition*. Journal of Functional Analysis, 268(6):1309 – 1381, 2015. (Cited on pages 10, 14, 22, 67, and 87.)

[Bouchut, 2002] F Bouchut. *Hypoelliptic regularity in kinetic equations*. Journal de Mathématiques Pures et Appliquées, 81(11):1135 – 1159, 2002. (Cited on pages 74, 80, and 81.)

[Bouleau and Hirsch, 1989] N. Bouleau and F. Hirsch. *On the derivability, with respect to the initial data, of the solution of a stochastic differential equation with Lipschitz coefficients*, pages 39–57. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989. (Cited on pages 10, 45, and 46.)

[Brezis, 2010] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 1st edition, 2010. (Cited on pages 65, 66, 75, 80, and 89.)

[Carrillo, 1998] José A. Carrillo. *Global weak solutions for the initial-boundary-value problems Vlasov-Poisson-Fokker-Planck System*. Mathematical Methods in the Applied Sciences, 21(10):907–938, 1998. (Cited on pages 74 and 79.)

[Costantini, 1991] C. Costantini. *Diffusion Approximation for a Class of Transport Processes with Physical Reflection Boundary Conditions*. Ann. Probab., 19(3):1071–1101, 1991. (Cited on page 14.)

[Costantini, 1992] C. Costantini. *The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations*. Probability Theory and Related Fields, 91(1):43–70, 1992. (Cited on page 14.)

[Darling and Pardoux, 1997] R. W. R. Darling and Etienne Pardoux. *Backwards SDE with random terminal time and applications to semilinear elliptic PDE*. Ann. Probab., 25(3):1135–1159, 1997. (Cited on page 87.)

[Dreeben and Pope, 1997] Thomas D. Dreeben and Stephen B. Pope. *Wall-function treatment in pdf methods for turbulent flows*. Physics of Fluids, 9(9):2692–2703, 1997. (Cited on page 10.)

[Einstein, 1905] A. Einstein. *Investigations On The Theory Of The Brownian Movement*. Ann. der Physik, 1905. (Cited on page 8.)

[Friedlander, 1977] S.K. Friedlander. *Smoke, dust, and haze: fundamentals of aerosol behavior*. Topicals in chemical engineering. Wiley, 1977. (Cited on page 7.)

[Friedman, 2012] A. Friedman. *Stochastic Differential Equations and Applications*. Dover Books on Mathematics. Dover Publications, 2012. (Cited on pages 20, 52, and 87.)

[Geman and Horowitz, 1973] D. Geman and J. Horowitz. *Occupation Times for Smooth Stationary Processes*. The Annals of Probability, 1(1):131–137, 1973. (Cited on page 70.)

[Gilbarg and Trudinger, 2001] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. U.S. Government Printing Office, 2001. (Cited on pages 70 and 85.)

[Giles, 2008] Michael B. Giles. *Multi-level Monte Carlo path simulation*. Operations Research, 56(3):607–617, 2008. (Cited on page 120.)

[Gobet, 2000] Emmanuel Gobet. *Weak approximation of killed diffusion using Euler schemes*. Stochastic Processes and their Applications, 87(2):167 – 197, 2000. (Cited on pages 92 and 100.)

[Gobet and Menozzi, 2004] Emmanuel Gobet and Stéphane Menozzi. *Exact approximation rate of killed hypoelliptic diffusions using the discrete Euler scheme*. Stochastic Processes and their Applications, 112(2):201 – 223, 2004. (Cited on page 92.)

[Hagan *et al.*, 1989] Patrick S. Hagan, Charles R. Doering, and C. David Levermore. *the distribution of exit times for weakly colored noise*. Journal of Statistical Physics, 54(5):1321–1352, 1989. (Cited on page 138.)

[Hecht, 2012] Frédéric Hecht. *New development in FreeFem++*. J. Numer. Math., 20(3-4):251–265, 2012. (Cited on page 101.)

[Heinrich, 1998] S. Heinrich. *Monte Carlo complexity of global solution of integral equations*. J. Complexity, 14(2):151–175, 1998. (Cited on page 119.)

[Henry *et al.*, 2014] Christophe Henry, Jean-Pierre Minier, Mikaël Mohaupt, Christophe Profeta, Jacek Pozorski, and Anne Tanière. *A stochastic approach for the simulation of collisions between colloidal particles at large time steps*. International Journal of Multiphase Flow, 61(0):94 – 107, 2014. (Cited on page 7.)

[Jacob, 2012] Emmanuel Jacob. *A Langevin process reflected at a partially elastic boundary: I*. Stochastic Processes and their Applications, 122(1):191 – 216, 2012. (Cited on page 14.)

[Jacob, 2013] Emmanuel Jacob. *A Langevin process reflected at a partially elastic boundary: II*. Lecture Notes in Mathematics, vol 2078. Springer Berlin Heidelberg, 2013. (Cited on page 14.)

[Karatzas and Shreve, 1991] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer New York, 1991. (Cited on page 138.)

[Kebaier, 2005] Ahmed Kebaier. *Statistical Romberg extrapolation: A new variance reduction method and applications to option pricing*. Ann. Appl. Probab., 15(4):2681–2705, 2005. (Cited on page 120.)

[Krylov, 1980] N.V. Krylov. *Controlled Diffusion Processes*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 1980. (Cited on page 31.)

[Ladyzhenskaia and Ural'ceva, 1968] O.A. Ladyzhenskaia and N.N. Ural'ceva. *Linear and Quasilinear Elliptic Equations*. Mathematics in Science and Engineering : a series of monographs and textbooks. Academic Press, 1968. (Cited on page 92.)

[Langevin, 1908] Paul Langevin. *Sur la théorie du mouvement brownien*, volume 146. 1908. (Cited on page 8.)

[Lépingle, 1995] D. Lépingle. *Euler scheme for reflected stochastic differential equations*. Mathematics and Computers in Simulation, 38(1):119 – 126, 1995. (Cited on pages 92 and 112.)

[Lions and Magenes, 1972] J.L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*. Number v. 3 in Non-homogeneous Boundary Value Problems and Applications. Springer-Verlag, 1972. (Cited on page 74.)

[Lions and Sznitman, 1984] P. L. Lions and A. S. Sznitman. *Stochastic differential equations with reflecting boundary conditions*. Communications on Pure and Applied Mathematics, 37(4):511–537, 1984. (Cited on page 14.)

[McKean, 1962] H. P. Jr. McKean. *A winding problem for a resonator driven by a white noise*. J. Math. Kyoto Univ., 2(2):227–235, 1962. (Cited on pages 8 and 14.)

[Meyer and Deglon, 2011] C.J. Meyer and D.A. Deglon. *Particle collision modeling - A review*. Minerals Engineering, 24(8):719 – 730, 2011. (Cited on page 7.)

[Minier and Peirano, 2001] Jean-Pierre Minier and Eric Peirano. *The PDF approach to turbulent polydispersed two-phase flows*. Physics Reports, 352(1–3):1 – 214, 2001. (Cited on page 8.)

[Minier and Pozorski, 1999] Jean-Pierre Minier and Jacek Pozorski. *Wall-boundary conditions in probability density function methods and application to a turbulent channel flow*. Physics of Fluids, 11(9):2632–2644, 1999. (Cited on page 10.)

[Nelson, 1967] Edward Nelson. *Dynamical Theories of Brownian Motion - Second Edition*. Princeton University Press, 1967. (Cited on page 138.)

[Nezza *et al.*, 2012] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. *Hitchhiker's guide to the fractional Sobolev spaces*. Bulletin des Sciences Mathématiques, 136(5):521 – 573, 2012. (Cited on page 81.)

[Pagès, 2017] Gilles Pagès. *Introduction To Numerical Probability applied to Finance*. 2017. (Cited on pages 91 and 95.)

[Paoli and Schatzman, 1993] L. Paoli and M. Schatzman. *Mouvement à un nombre fini de degrés de liberté avec contraintes unilatérales : cas avec perte d'énergie*. ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, 27(6):673–717, 1993. (Cited on pages 14, 92, and 113.)

[Pardoux and Veretennikov, 2001] E. Pardoux and Yu. Veretennikov. *On the Poisson Equation and Diffusion Approximation. I.* Ann. Probab., 29(3):1061–1085, 2001. (Cited on page 138.)

[Pavliotis, 2014] G.A. Pavliotis. *Stochastic Processes and Applications: Diffusion Processes, the Fokker-Planck and Langevin Equations*. Texts in Applied Mathematics. Springer New York, 2014. (Cited on pages 137 and 138.)

[Rana, 2005] I.K. Rana. *An Introduction to Measure and Integration*. Alpha Science international, 2005. (Cited on page 74.)

[Slominski, 2013] Leszek Slominski. *Weak and strong approximations of reflected diffusions via penalization methods*. Stochastic Processes and their Applications, (123):752–763, 2013. (Cited on pages 92 and 115.)

[von Smoluchowski, 1917] Marian von Smoluchowski. *Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen*. Zeitschrift fuer physikalische Chemie, 92:129 – 168, 1917. (Cited on page 7.)

[Spiliopoulos, 2007] Konstantinos Spiliopoulos. *A Note on the Smoluchowski-Kramers Approximation Langevin Equation with Reflection*. Stochastics and Dynamics, 07(02):141–152, 2007. (Cited on pages 14 and 138.)

[Talay and Tubaro, 1989] Denis Talay and Luciano Tubaro. *Expansion of the global error for numerical schemes solving stochastic differential equations*. Research Report RR-1069, INRIA, 1989. (Cited on page 91.)

[Tartar, 1978] L. Tartar. *Topics in nonlinear analysis*. Publications mathématiques d'Orsay. Université de Paris-Sud, Dép. de mathématique, 1978. (Cited on page 74.)

[Ye *et al.*, 2007] Haiping Ye, Jianming Gao, and Yongsheng Ding. *A generalized Gronwall inequality and its application to a fractional differential equation*. Journal of Mathematical Analysis and Applications, 328(2):1075 – 1081, 2007. (Cited on page 189.)